

On the Transience of Linear Max-Plus Dynamical Systems

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Abstract

We study the transients of linear max-plus dynamical systems. For that, we consider for each irreducible max-plus matrix A , the weighted graph $G(A)$ such that A is the adjacency matrix of $G(A)$. Based on a novel graph-theoretic counterpart to the number-theoretic Brauer's theorem, we propose two new methods for the construction of arbitrarily long paths in $G(A)$ with maximal weight. That leads to two new upper bounds on the transient of a linear max-plus system which both improve on the bounds previously given by Even and Rajsbaum (STOC 1990, Theory of Computing Systems 1997), by Bouillard and Gaujal (Research Report 2000), and by Soto y Koelemeijer (PhD Thesis 2003), and are, in general, incomparable with Hartmann and Arguelles' bound (Mathematics of Operations Research 1999). With our approach, we also show how to improve the latter bound by a factor of two.

A significant benefit of our bounds is that each of them turns out to be linear in the size of the system in various classes of linear max-plus system whereas the bounds previously given are all at least quadratic. Our second result concerns the relationship between matrix and system transients: We prove that the transient of an $N \times N$ matrix A is, up to some constant, equal to the transient of an A -linear system with an initial vector whose norm is quadratic in N . Finally, we study the applicability of our results to the well-known Full Reversal algorithm whose behavior can be described as a min-plus linear system.

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1 Introduction

The mathematical theory of linear max-plus dynamical systems provides tools to understand the complex behavior of many important distributed systems. Of particular interest are transportation and automated manufacturing systems [1, 2, 3], or network synchronizers [4]. As shown by Charron-Bost et al. [5], another striking example is the *Full Reversal* distributed algorithm which can be used to solve a variety of problems like routing [6] and scheduling [7]. The fundamental theorem in max-plus algebra—an analog of the Perron-Frobenius theorem—states that the powers of an irreducible max-plus matrix become periodic after a finite index, called the *transient* of the matrix (see for instance [8]). As an immediate corollary, any linear max-plus system is periodic from some index, called the *transient* of the (linear) system, which clearly depends on initial conditions and is at most equal to the transient of the matrix of the system.

For all the above mentioned applications, the study of the transient plays a key role in characterizing their performances: For example, in case of Full Reversal routing, the system transient corresponds to the time until the routing algorithm terminates in a destination oriented routing graph. Besides that, understanding the matrix and system transient is of interest on its own for the theory of max-plus algebra. While the transients of matrices and linear systems have been shown to be computable in polynomial time by Hartmann and Arguelles [9], their algorithms provide no analysis of the transient phase, as they both use (binary) search at heart, and by that do not hint at the parameters that influence matrix and system transients. Conversely, upper bounds involving these parameters would help to predict the duration of the transient phase, and to define strategies to reduce the transient as well. Hence concerning transience bounds, our contribution is *both* numerical and methodological.

The problem of bounding the transients has already been studied: Bouillard and Gaujal [10] have given an upper bound on the transient of a matrix which is exponential in the size of the matrix, and polynomial bounds have been established by Even and Rajsbaum [4] for linear systems with integer coefficients, by Soto y Koelemeijer [11] for both general matrices and linear systems, and by Hartmann and Arguelles [9] for general matrices and linear systems in max-plus algebra. In each of these works, the problem of studying the transient is reduced to the study of paths in a specific graph: For every max-plus matrix A , one considers the weighted directed graph G whose adjacency matrix is A , and the *critical subgraph* of G which consists of the *critical closed paths* in G , namely those closed paths with maximal average weight. The periodic behavior of the powers of A is intimately related to the structure of the critical subgraph of G : Bounding transients amounts to bounding the weights of arbitrary long paths in the graph. The first step in controlling the weights of paths consists in reaching the critical subgraph with sufficiently long paths. With respect to this first step, the methods used in the four above-mentioned transience bounds are rather similar. The approaches mainly differ in the way the critical subgraph is then visited.

In this article, we propose two new methods, namely the *explorative method* and the *repetitive method* for visiting the critical subgraph. The first one consists in exploring the whole strongly connected components of the critical subgraph whereas in the second one, the visit of the critical subgraph is confined to repeatedly follow only one closed path. That leads us to two new upper bounds on the transients of linear systems which improve on the bounds given by Even and Rajsbaum and by Soto y Koelemeijer, and are incomparable with Hartmann and Arguelles' bound, for which we show how to improve it by a factor of two.

A significant benefit of our bounds lies in the fact that each of them turns out to be linear in the size of the system (i.e., the number of nodes) in some important graph families (e.g., trees) whereas

the bounds previously given are all at least quadratic. This is mainly due to the introduction of new graph parameters that enable a fine-grained analysis of the transient phase. In particular, we introduce the notion of the *exploration penalty* of a graph G as the least integer k with the property that, for every $n \geq k$ divisible by the cyclicity of G and every node i of G , there is a closed path starting and ending at i of length n . One key point is then an at most quadratic upper bound on the exploration penalty which we derive from the number-theoretic Brauer's Theorem [12].

Another contribution of this paper concerns the relationship between matrix and system transients: We prove that the transient of an $N \times N$ matrix A —which is clearly an upper bound on all transients of A -linear systems—is, up to some constant, equal to the transient of an A -linear system with an initial vector whose norm is quadratic in N . In addition to shedding new light on transients, this result provides a direct method for deriving upper bounds on matrix transients from upper bounds on system transients.

The paper is organized as follows. Section 2 introduces basic notions of graph theory and max-plus algebra. We show an upper bound on lengths of maximum weight paths that do not visit the critical subgraph in Section 3. In Section 4, we introduce the notion of *exploration penalty* and improve a theorem by Denardo [13] on the existence of arbitrarily long paths in strongly connected graphs. Sections 5 and 6 introduce our explorative and repetitive bounds, respectively. We show how to convert upper bounds on the transients of max-plus systems to upper bounds on the transients of max-plus matrices in Section 7. We discuss our results, by comparing them to previous work and by applying them to the analysis of the Full Reversal algorithm, in Section 8.

2 Preliminaries

2.1 Basic definitions

Denote by \mathbb{N} the set of nonnegative integers and let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$. In this paper, we follow the convention $\max \emptyset = -\infty$.

A (*directed*) *graph* G is a pair (V, E) , where V is a nonempty finite set and $E \subseteq V \times V$. We call the elements of V the *nodes* of G and the elements of E the *edges* of G . An edge $e = (i, j)$ is called *incident to* node k if $k = i$ or $k = j$. We say that G is *nontrivial* if E is nonempty.

A graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. For a nonempty subset E' of E , let the *subgraph of G induced by edge set E'* be the graph (V', E') where $V' = \{i \in V \mid \exists j \in V : (i, j) \in E' \vee (j, i) \in E'\}$.

A *path* π in G is a triple $\pi = (\text{Start}, \text{Edges}, \text{End})$ where Start and End are nodes in G , Edges is a sequence (e_1, e_2, \dots, e_n) of edges $e_l = (i_l, j_l)$ such that $j_l = i_{l+1}$ if $1 \leq l \leq n-1$ and, if Edges is nonempty, $i_1 = \text{Start}$ and $j_n = \text{End}$, and if Edges is empty, $\text{Start} = \text{End}$. We say that π is *empty* if Edges is empty. We define the operators Start , Edges , and End on the set of paths by setting $\text{Start}(\pi) = \text{Start}$, $\text{Edges}(\pi) = \text{Edges}$, and $\text{End}(\pi) = \text{End}$. Define the *length* $\ell(\pi)$ of π as the length of $\text{Edges}(\pi)$.

Let $\mathcal{P}(i, j, G)$ denote the set of paths π in G with $\text{Start}(\pi) = i$ and $\text{End}(\pi) = j$, and by $\mathcal{P}(i \rightarrow, G)$ the set of paths π in G with $\text{Start}(\pi) = i$. If $\pi \in \mathcal{P}(i, j, G)$, we say that i is the *start node* of π and j is the *end node* of π . We write $\mathcal{P}^n(i, j, G)$ (respectively $\mathcal{P}^n(i \rightarrow, G)$), where $n \geq 0$, for the set of paths in $\mathcal{P}(i, j, G)$ (respectively $\mathcal{P}(i \rightarrow, G)$) of length n . Path π is *closed* if $\text{Start}(\pi) = \text{End}(\pi)$. Let $\mathcal{P}_\circ(G)$ denote the set of *nonempty* closed paths in graph G .

Path π is *elementary* if the nodes i_l are pairwise distinct. Intuitively, an elementary path visits

at most one node twice: its start node. All elementary paths π satisfy $\ell(\pi) \leq |V|$. A path is *simple* if it is elementary and non-closed, or empty. Intuitively, a path is simple if it does not visit the same node twice. All simple paths π satisfy $\ell(\pi) \leq |V| - 1$.

Call G *strongly connected* if $\mathcal{P}(i, j, G)$ is nonempty for all nodes i and j of G . A subgraph H of G is called a *strongly connected component* of G if H is maximal with respect to the subgraph relation such that H is strongly connected. Denote by $\mathcal{C}(G)$ the set of strongly connected components of G . For every node i of G there exists exactly one H in $\mathcal{C}(G)$ such that i is a node of H .

For two paths π and π' in G , we say that π' is a *prefix* of π if $\text{Start}(\pi) = \text{Start}(\pi')$ and $\text{Edges}(\pi')$ is a prefix of $\text{Edges}(\pi)$. We say that π' is a *postfix* of π if $\text{End}(\pi) = \text{End}(\pi')$ and $\text{Edges}(\pi')$ is a postfix of $\text{Edges}(\pi)$. We call π' a *subpath* of π if it is the postfix of some prefix of π . We say a node i is a *node of path* π if there exists a prefix π' of π with $\text{End}(\pi') = i$, and an edge e is an *edge of path* π if e occurs within $\text{Edges}(\pi)$. For two paths π_1 and π_2 with $\text{End}(\pi_1) = \text{Start}(\pi_2)$, define the concatenation $\pi = \pi_1 \cdot \pi_2$ by setting $\text{Start}(\pi) = \text{Start}(\pi_1)$, $\text{Edges}(\pi) = \text{Edges}(\pi_1) \cdot \text{Edges}(\pi_2)$ and $\text{End}(\pi) = \text{End}(\pi_2)$.

We define the *girth* $g(G)$ and *circumference* $cr(G)$ of G to be the minimum resp. maximum length of nonempty elementary closed paths in G . Define the *cab driver's diameter* $cd(G)$ of G to be the maximum length of simple paths in G . The *cyclicity* of G is defined as

$$c(G) = \text{lcm} \{ \gcd\{\ell(\gamma) \mid \gamma \in \mathcal{P}_\circ(H)\} \mid H \in \mathcal{C}(G) \} .$$

For a strongly connected graph G , the cyclicity $c(G)$ is the greatest common divisor of closed path lengths in G ; in particular $c(G) \leq g(G) \leq |V|$. We say that G is *primitive* if $c(G) = 1$. A graph is primitive if and only if all its strongly connected components are primitive. We also define the two graph parameters

$$d(G) = \max\{c(H) \mid H \in \mathcal{C}(G)\}$$

and

$$p(G) = \text{lcm}\{\ell(\gamma) \mid \gamma \in \mathcal{P}_\circ(H) \wedge \gamma \text{ is elementary}\} .$$

It is $d(G) \leq c(G) \leq p(G)$.

Lemma 1. *Let π be a path in G . Then there exist paths π_1, π_2 and an elementary closed path γ such that $\pi = \pi_1 \cdot \gamma \cdot \pi_2$. Moreover, if π is non-simple, then γ can be chosen to be nonempty.*

Proof. The first claim is trivial, for we can choose $\pi_1 = \pi$, and γ and π_2 to be empty.

So let π be non-simple. Then π is necessarily nonempty. If π is elementary closed, set $\gamma = \pi$ and choose π_1 and π_2 to be empty. Otherwise, let $\text{Edges}(\pi) = (e_1, \dots, e_n)$ and $e_l = (i_l, j_l)$. By assumption, there exist $k < l$ such that $i_k = i_l$ and $l - k$ is minimal. Now set $\text{Edges}(\pi_1) = (e_1, \dots, e_{k-1})$, $\text{Edges}(\gamma) = (e_k, \dots, e_{l-1})$, $\text{Edges}(e_l, \dots, e_n)$, and choose the start and end nodes accordingly. Closed path γ is elementary, because $l - k$ was chosen to be minimal. \square

An *edge-weighted* (or *e-weighted*) graph G is a triple (V, E, w_E) such that (V, E) is a graph and $w_E : E \rightarrow \mathbb{R}$. An *edge-node-weighted* (or *en-weighted*) graph G is a quadruple (V, E, w_E, w_V) such that (V, E, w_E) is an e-weighted graph and $w_V : V \rightarrow \mathbb{R}_{\max}$.

If $G = (V, E, w_E, w_V)$ is an en-weighted graph and π is a path in G with $\text{Edges}(\pi) = (e_1, \dots, e_n)$, we define the *en-weight* of π as

$$w(\pi) = \sum_{l=1}^n w_E(e_l) + w_V(\text{End}(\pi)) ,$$

and, if G is just an e-weighted graph, the e -weight of π as

$$w_*(\pi) = \sum_{l=1}^n w_E(e_l) .$$

Let $w^n(i \rightarrow, G) = \max\{w(\pi) \mid \pi \in \mathcal{P}^n(i \rightarrow, G)\}$.

In the rest of the paper, every notion introduced for graphs (resp. e-weighted graphs) is trivially extended to e-weighted graphs (resp. en-weighted graphs) using the same terminology and notation.

2.2 Realizers

Let \mathbf{N} be any nonempty set of nonnegative integers, and let i be any node of a strongly connected e-weighted graph G . A path $\tilde{\pi}$ is said to be an \mathbf{N} -realizer for node i if $\tilde{\pi} \in \mathcal{P}(i \rightarrow, G)$, $\ell(\tilde{\pi}) \in \mathbf{N}$, and $w(\tilde{\pi}) = \sup_{n \in \mathbf{N}} w^n(i \rightarrow)$. Obviously, an \mathbf{N} -realizer exists for any node if \mathbf{N} is finite. Of particular interest is the case of sets \mathbf{N} of the form

$$\mathbf{N}_{\geq \hat{n}}^{(r,p)} = \{n \in \mathbb{N} \mid n \geq \hat{n} \wedge n \equiv r \pmod{p}\} ,$$

where $r \in \mathbb{N}$, and $p, \hat{n} \in \mathbb{N}^*$. The following lemma gives a useful sufficient condition in terms of $\mathbf{N}_{\geq \hat{n}}^{(r,p)}$ -realizer to guarantee the eventual periodicity of the sequence $(w^n(i \rightarrow))_{n \in \mathbb{N}}$.

Lemma 2. *Let i be a node and let p and \hat{n} be positive integers. Suppose that for all $n \geq \hat{n}$, there exists a $\mathbf{N}_{\geq \hat{n}}^{(n,p)}$ -realizer for i of length n . Then $w^{n+p}(i \rightarrow) = w^n(i \rightarrow)$ for all $n \geq \hat{n}$.*

Proof. For each integer $n \geq \hat{n}$, let π_n be one $\mathbf{N}_{\geq \hat{n}}^{(n,p)}$ -realizer for i of length n . Denote by $X(n)$ the set of paths π in $\mathcal{P}(i \rightarrow)$ that satisfy $\ell(\pi) \equiv n \pmod{p}$ and $\ell(\pi) \geq \hat{n}$, and let $x(n)$ be the supremum of values $w(\pi)$ where $\pi \in X(n)$. Since $\mathbf{N}_{\geq \hat{n}}^{(n,p)}$ has a realizer, each $x(n)$ is finite, and $x(n) = w(\pi_n)$.

From $n + p \equiv n \pmod{p}$, it follows

$$X(n + p) = X(n)$$

and so $x(n + p) = x(n)$. For all $n \geq \hat{n}$, we have $\mathcal{P}^n(i \rightarrow) \subseteq X(n)$, i.e., $w^n(i \rightarrow) \leq x(n)$. As $n + p > n$, we also conclude that $w^{n+p}(i \rightarrow) \leq x(n + p)$. Because $\pi_n \in \mathcal{P}^n(i \rightarrow)$, we have $w^n(i \rightarrow) \geq w(\pi_n)$. Similarly, $w^{n+p}(i \rightarrow) \geq w(\pi_{n+p})$. This concludes the proof. \square

2.3 The critical subgraph

Let $G = (V, E, w_E)$ be a nontrivial strongly connected e-weighted graph. Define the *rate* $\varrho(G)$ of G by

$$\varrho(G) = \sup \left\{ \frac{w_*(\gamma)}{\ell(\gamma)} \mid \gamma \in \mathcal{P}_{\circ}(G) \right\} ,$$

which is easily seen to be finite. A (nonempty) closed path $\gamma \in \mathcal{P}_{\circ}(G)$ is *critical* if $w_*(\gamma)/\ell(\gamma) = \varrho(G)$. A node of G is *critical* if it is node of a critical path in G , and an edge of G is *critical* if it is an edge of a critical path in G . The *critical subgraph* of G , denoted by G_c , is the subgraph of G induced by the set of critical edges of G . A *critical component* of G is a strongly connected component of the critical subgraph of G .

We denote by $\Delta(G)$ (respectively $\delta(G)$) the maximum (respectively minimum) edge weights in G . Let $\Delta_{nc}(G)$ denote the maximum weight of edges between two non-critical nodes. If no

such edge exists, set $\Delta_{\text{nc}}(G) = \varrho(G)$. Note that $\delta(G) \leq \varrho(G) \leq \Delta(G)$ always holds. Denote the *non-critical rate of G* by

$$\varrho_{\text{nc}}(G) = \sup \left\{ \frac{w_*(\gamma)}{\ell(\gamma)} \mid \gamma \in \mathcal{P}_{\circlearrowleft}(G) \wedge \gamma \text{ has no critical node} \right\} ,$$

with the classical convention that $\varrho_{\text{nc}}(G) = -\infty$. if no such path exists.

We now study how the critical graph and the various parameters introduced above are modified by homotheties. Let λ be any element in \mathbb{R} , and let $\lambda \otimes G$ denote the e -weighted graph $\lambda \otimes G = (V, E, \lambda \otimes w_E)$ where for any edge $e \in E$

$$\lambda \otimes w_E(e) = w_E(e) + \lambda .$$

Lemma 3. *The e -weighted graph $\lambda \otimes G$ has the same critical subgraph as G , and its rate is equal to $\varrho(\lambda \otimes G) = \varrho(G) + \lambda$.*

Proof. If $w_*(\pi)$ and $\lambda \otimes w_*(\pi)$ denote the respective e -weights of path π in G and $\lambda \otimes G$, then we have the equality $\lambda \otimes w_*(\pi)/\ell(\pi) = w_*(\pi)/\ell(\pi) + \lambda$, which implies $\varrho(\lambda \otimes G) = \varrho(G) + \lambda$. The equality $(\lambda \otimes G)_c = G_c$ now easily follows. \square

From Lemma 3, we easily check that each of the parameters $\Delta(G) - \varrho(G)$, $\Delta_{\text{nc}}(G) - \varrho(G)$, and $\delta(G) - \varrho(G)$ is invariant under homothety:

Lemma 4. *Let G be an e -weighted graph, and λ an element in \mathbb{R} . If $\lambda \otimes G$ denotes the e -weighted graph obtained by adding λ to each edge weight of G , then $\Delta(\lambda \otimes G) - \varrho(\lambda \otimes G) = \Delta(G) - \varrho(G)$, $\Delta_{\text{nc}}(\lambda \otimes G) - \varrho(\lambda \otimes G) = \Delta_{\text{nc}}(G) - \varrho(G)$, and $\delta(\lambda \otimes G) - \varrho(\lambda \otimes G) = \delta(G) - \varrho(G)$.*

We next show that critical closed paths exist by showing that $\varrho(G)$ is equal to the supremum of the finite set of $w_*(\gamma)/\ell(\gamma)$ where γ is a nonempty *elementary* closed path. Define

$$\varrho_e(G) = \sup \left\{ \frac{w_*(\gamma)}{\ell(\gamma)} \mid \gamma \in \mathcal{P}_{\circlearrowleft}(G) \wedge \gamma \text{ is elementary} \right\} .$$

Lemma 5. *Let a, b, c, d be real numbers, b and d positive, such that $a/b \leq c/d$. Then:*

$$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$$

Proof. We have $ad \leq bc$ and thus

$$\frac{a+c}{b+d} = \frac{1}{b} \cdot \frac{ab+bc}{b+d} \geq \frac{1}{b} \cdot \frac{ab+ad}{b+d} = \frac{a}{b} .$$

Analogously,

$$\frac{a+c}{b+d} = \frac{1}{d} \cdot \frac{ad+cd}{b+d} \leq \frac{1}{d} \cdot \frac{bc+cd}{b+d} = \frac{c}{d} ,$$

which concludes the proof. \square

Lemma 6. *For every nontrivial strongly connected e -weighted graph G ,*

$$\varrho(G) = \varrho_e(G) .$$

In particular, there exists an elementary critical closed path in G .

Proof. Obviously, $\varrho_e(G) \leq \varrho(G)$.

Conversely, we show by induction on $\ell(\gamma)$ that $w_*(\gamma)/\ell(\gamma) \leq \varrho_e(G)$ for all nonempty closed paths γ in G . The case $\ell(\gamma) = 1$ is trivial, because every closed path of length 1 is elementary. Now let $\ell(\gamma) > 1$. If γ is elementary, we are done by the definition of $\varrho_e(G)$. If γ is non-elementary, it is non-simple. Thus, by Lemma 1, there exist π_1 , π_2 , and an elementary nonempty closed path γ' such that $\gamma = \pi_1 \cdot \gamma' \cdot \pi_2$. It is $\text{End}(\pi_1) = \text{Start}(\pi_2)$, hence $\gamma'' = \pi_1 \cdot \pi_2$ is a closed path. Furthermore, $\ell(\gamma') < \ell(\gamma)$ because $\gamma' \neq \gamma$, and $\ell(\gamma'') < \ell(\gamma)$ because γ' is nonempty.

We obtain

$$\frac{w_*(\gamma)}{\ell(\gamma)} = \frac{w_*(\gamma') + w_*(\gamma'')}{\ell(\gamma') + \ell(\gamma'')} \leq \max \left\{ \frac{w_*(\gamma')}{\ell(\gamma')} , \frac{w_*(\gamma'')}{\ell(\gamma'')} \right\} \leq \varrho_e(G)$$

by Lemma 5 and the induction hypothesis. Thus we have shown $\varrho(G) \leq \varrho_e(G)$.

The last statement follows because the set of nonempty elementary closed paths is finite. \square

The following is a well-known fact in max-plus algebra:

Lemma 7. *Let G be a nontrivial strongly connected e -weighted graph. Then every nonempty closed path in G_c is critical in G .*

Proof. Set $\varrho = \varrho(G)$. Let π be a nonempty closed path in G_c with $\text{Edges}(\pi) = (e_1, \dots, e_n)$. By definition of G_c , all edges $e_l = (i_l, j_l)$, $1 \leq l \leq n$, are critical, i.e., there exists a path $\gamma_l \in \mathcal{P}_{\circlearrowleft}(G)$ with

$$w_*(\gamma_l)/\ell(\gamma_l) = \varrho , \tag{1}$$

and whose last edge is e_l . For each l , $1 \leq l \leq n$, construct γ'_l from γ_l by removing its last edge. Thus γ'_l is a path with $\text{Start}(\gamma'_l) = j_l$ and $\text{End}(\gamma'_l) = i_l$. Concatenation of the paths γ'_l yields a closed path $\gamma' = \gamma'_n \cdot \gamma'_{n-1} \cdot \dots \cdot \gamma'_1$. From (1) follows

$$w_*(\gamma'_l) + w_E(e_l) = \varrho \ell(\gamma'_l) + \varrho .$$

Summing over all l yields

$$w_*(\gamma') + w_*(\pi) = \varrho \ell(\gamma') + \varrho \ell(\pi) .$$

Combination with $\varrho \ell(\gamma') \geq w_*(\gamma')$ gives

$$w_*(\pi) \geq \varrho \ell(\pi) ,$$

hence π is critical. \square

2.4 Linear max-plus systems

A matrix with entries in \mathbb{R}_{\max} is called a *max-plus* matrix. We denote by $\mathcal{M}_{M,N}(X)$ the set of $M \times N$ matrices with entries in X . If $A \in \mathcal{M}_{M,N}(\mathbb{R}_{\max})$ and $B \in \mathcal{M}_{N,Q}(\mathbb{R}_{\max})$, define $A \otimes B \in \mathcal{M}_{M,Q}(\mathbb{R}_{\max})$ by setting

$$(A \otimes B)_{i,j} = \max \{ A_{i,k} + B_{k,j} \mid 1 \leq k \leq N \} .$$

The identity element in the monoid $(\mathcal{M}_{N,N}(\mathbb{R}_{\max}), \otimes)$, denoted $[0]_N$, is the matrix whose diagonal entries are equal to 0 and all other entries are equal to $-\infty$. More generally, for any $\lambda \in \mathbb{R}_{\max}$, the matrix whose diagonal entries are equal to λ and all other entries are equal to $-\infty$ is denoted by $[\lambda]_N$. If $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$, define $A^{\otimes 0} = [0]_N$, and $A^{\otimes n} = A \otimes A^{\otimes n-1}$ for $n \geq 1$. For convenience, the matrix $[\lambda]_N \otimes A$ is simply written $\lambda \otimes A$; more generally, for any positive integer n , the matrix $([\lambda]_N)^{\otimes n} \otimes A$, whose (i, j) -entry is $A_{i,j} + \lambda n$, is simply written $\lambda^{\otimes n} \otimes A$.

For a matrix $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ and a vector $v \in \mathbb{R}_{\max}^N$, we define the *linear max-plus system* $x_{A,v}$ by setting

$$x_{A,v}(n) = \begin{cases} v & \text{for } n = 0 \\ A \otimes x_{A,v}(n-1) & \text{for } n \geq 1 \end{cases} . \quad (2)$$

Clearly $x_{A,v}(n) = A^{\otimes n} \otimes v$.

Denote by $G(A)$ the e-weighted graph with nodes $\{1, 2, \dots, N\}$ containing an edge (i, j) if and only if $A_{i,j}$ is finite, and set its edge weight $w_E(i, j) = A_{i,j}$. Observe that with the notation in the previous Section, $G(\lambda \otimes A) = \lambda \otimes G(A)$.

Denote by $G(A, v)$ the en-weighted graph defined in the same way as $G(A)$ with node weights set to $w_V(i) = v_i$. Call A *irreducible* if $G(A)$ is strongly connected and nontrivial.

The following correspondence between the sequence $(A^{\otimes n}(n))_{n \geq 0}$ (respectively the sequence $x_{A,v}$) and weights of paths in $G(A)$ (respectively $G(A, v)$) holds:

Lemma 8. *For all matrices $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ and all $v \in \mathbb{R}_{\max}^N$,*

$$(A^{\otimes n})_{i,j} = \max \{w_*(\pi) \mid \pi \in \mathcal{P}^n(i, j, G(A))\} \text{ and} \\ (A^{\otimes n} \otimes v)_i = \max \{w(\pi) \mid \pi \in \mathcal{P}^n(i \rightarrow, G(A, v))\} = w^n(i \rightarrow, G(A, v)) .$$

We simply denote $(A^{\otimes n})_{i,j}$ by $A_{i,j}^{\otimes n}$ in the sequel.

We define the *cyclicity* of A , denoted by $c(A)$, as the cyclicity of the critical subgraph of $G(A)$, i.e.,

$$c(A) = c(G_c(A)) .$$

The main result on max-plus matrices is an analog of the Perron-Frobenius theorem in linear algebra.

Theorem 1 ([8, Theorems 2.9 and 3.9]). *An irreducible matrix $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ has exactly one eigenvalue $\varrho(A)$ equal to the rate of the e-weighted graph $G(A)$, i.e.,*

$$\varrho(A) = \varrho(G(A)) .$$

Moreover, there exists an integer \hat{n} such that for every $n \geq \hat{n}$:

$$A^{\otimes n+c(A)} = (\varrho(A))^{\otimes c(A)} \otimes A^{\otimes n} .$$

Using the classical operators “+” and “.”, the above equality is equivalent to:

$$\forall i, j \in \{1, \dots, N\} : \quad A_{i,j}^{\otimes n+c(A)} = A_{i,j}^{\otimes n} + c(A)\varrho(A) .$$

2.5 Eventually periodic sequences

Let X be any nonempty set. A sequence $f : \mathbb{N} \rightarrow \mathbb{R}_{\max}^X$ is *eventually periodic* if there exist $p \in \mathbb{N}^*$, $w_p \in \mathbb{R}_{\max}$, and $n_p \in \mathbb{N}$ such that

$$\forall n \geq n_p : f(n+p) = f(n) + w_p ,$$

where w_p stands for the constant function that maps any element in X to w_p . Such an integer p is called an *eventual period* (or for short a *period*) of f . Theorem 1 shows that $c(A)$ is a period of both sequences $(A^{\otimes n})_{n \geq 0}$ and $(x_{A,v}(n))_{n \geq 0}$.

We denote by \mathbf{P}_f the set of periods of f . Clearly \mathbf{P}_f is a nonempty subset of \mathbb{N} closed under addition. Let $p_0 = \min \mathbf{P}_f$ be the minimal period of f ; hence $p_0 \mathbb{N}^* \subseteq \mathbf{P}_f$. As $p_0 \in \mathbf{P}_f$, there exist $w_0 \in \mathbb{R}_{\max}$ and $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 : f(n+p) = f(n) + w_0 .$$

Let p be any period of f , and $p = ap_0 + b$ the Euclidean division of p by p_0 . For any integer $n \geq \max\{n_p, n_0 - b\}$,

$$f(n+p) = f(n) + w_p = f(n+b) + aw_0 .$$

It follows that either $b = 0$ or b is a period of f . Since $b \leq p_0 - 1$ and p_0 is the smallest period of f , we have $b = 0$, i.e., p_0 divides p . Then we derive that $\mathbf{P}_f \subseteq p_0 \mathbb{N}^*$.

For any period $p = qp_0$ of f , let n_q be the smallest positive integer such that

$$\forall n \geq n_q : f(n+qp_0) = f(n) + qw_0 .$$

The integer n_q is called the *transient of f for period p* .

Lemma 9. *For any positive integer q , $n_q = n_1$.*

Proof. Since for any $n \geq n_1$,

$$f(n+qp_0) = f(n) + qw_0,$$

we have $n_q \leq n_1$.

We now prove that $n_q = n_1$ by induction on $q \in \mathbb{N}^*$.

1. The base case $q = 1$ is trivial.
2. Assume that $n_q = n_1$. For any integer $n \geq n_{q+1}$,

$$f(n+(q+1)p_0) = f(n) + (q+1)w_0.$$

Moreover, if $n + p_0 \geq n_q$ then

$$f(n+(q+1)p_0) = f(n+p_0) + pw_0.$$

It follows that for any integer $n \geq \max\{n_q - p_0, n_{q+1}\}$,

$$f(n+p_0) = f(n) + w_0.$$

Hence $n_1 \leq \max\{n_q - p_0, n_{q+1}\}$, and by inductive assumption $n_1 \leq \max\{n_1 - p_0, n_{q+1}\}$. Then we derive $n_1 \leq n_{q+1}$, and so $n_1 = n_{q+1}$ as required. \square

It follows that the transient for period p of a periodic sequence is independent of p . We hence simply call it the *transient of f* .

Theorem 1 states that $c(A) = c(G_c(A))$ is a period of the sequence $A^{\otimes n}$. Because $p(A) = p(G(A))$ is a multiple of $c(A)$, also $p(A)$ is a period. We find it more convenient in our proofs to consider $p(A)$ as the period instead of the period $c(A)$ suggested by Theorem 1.

Let $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ be an irreducible matrix and let $v \in \mathbb{R}_{\max}^N$. We call the transient of the sequence $(A^{\otimes n})_{n \geq 0}$ the *transient of matrix A* , denoted by n_A , and the transient of the sequence $(x_{A,v}(n))_{n \geq 0}$ the *transient of system $x_{A,v}$* , denoted by $n_{A,v}$. Obviously, n_A is an upper bound on the transient of the $x_{A,v}$'s, i.e.,

$$\sup \{n_{A,v} | v \in \mathbb{R}_{\max}^N\} \leq n_A .$$

Conversely, the equalities

$$A_{i,j}^{\otimes n} = (A^{\otimes n} \otimes e^j)_i$$

where $e_i^j = 0$ if $i = j$ and $e_i^j = -\infty$ otherwise, show that

$$\max \{n_{A,e^j} | j \in \{1, \dots, N\}\} \geq n_A .$$

Hence,

$$\sup \{n_{A,v} | v \in \mathbb{R}_{\max}^N\} = n_A . \quad (3)$$

The transients are invariant under homotheties:

Lemma 10. *For all irreducible matrices $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$, all vectors $v \in \mathbb{R}_{\max}^N$, and all $\lambda \in \mathbb{R}$, we have the equalities of transients $n_{\lambda \otimes A} = n_A$ and $n_{\lambda \otimes A, v} = n_{A,v}$.*

Proof. The lemma follows immediately from the equalities

$$(\lambda \otimes A)^{\otimes n} = \lambda^{\otimes n} \otimes A^{\otimes n}$$

and

$$(\lambda \otimes A)^{\otimes n} \otimes v = \lambda^{\otimes n} \otimes (A^{\otimes n} \otimes v) .$$

□

2.6 Reduction to the case of a zero rate

Let $G = (V, E, w_E)$ be a nontrivial strongly connected e-weighted graph. Since $\varrho(G) \in \mathbb{R}$, we may define the e-weighted graph $\overline{G} = (-\varrho(G)) \otimes G$. By Lemma 3, G and \overline{G} have the same critical subgraph, and

$$\varrho(\overline{G}) = 0 .$$

Similarly, for any irreducible max-plus matrix A , we denote $\overline{A} = (-\varrho(A)) \otimes A$, and we have

$$\varrho(\overline{A}) = 0 .$$

Moreover, Lemma 4 gives:

$$\Delta(\overline{G}) = \Delta(G) - \varrho(G), \quad \Delta_{\text{nc}}(\overline{G}) = \Delta_{\text{nc}}(G) - \varrho(G), \quad \delta(\overline{G}) = \delta(G) - \varrho(G) ,$$

which are respectively denoted $\overline{\Delta}(G)$, $\overline{\Delta}_{\text{nc}}(G)$, and $\overline{\delta}(G)$. From $\varrho(\overline{G}) = 0$, we easily deduce that

$$\overline{\delta}(G) \leq 0 \leq \overline{\Delta}(G) .$$

The point of the reduction to a zero rate is evidenced by the following lemma:

Lemma 11. *Let \mathbf{N} be any nonempty set of nonnegative integers, and let i be any node of a strongly connected e -weighted graph G such that $\varrho(G) = 0$. Then there exists an \mathbf{N} -realizer for node i .*

Proof. By Theorem 1 and the fact that $\varrho(G) = 0$, the supremum is taken over a finite set. Hence it is a maximum. \square

3 Visiting the Critical Subgraph along Optimal Paths

In this section, we give an upper bound B_c on lengths of paths with maximum weight containing no critical node. For that, we first describe how to extract a simple path from an arbitrary path.

3.1 Our first path reduction

In this section, we construct from a path π its simple part $\text{Simp}(\pi)$ by repeatedly removing nonempty closed subpaths. Each step of this construction corresponds to applying Lemma 1. As the decomposition in this lemma is not unique, we choose the to-be-removed closed subpath non-deterministically. Formally, we fix a global choice function¹ which we use every time we “choose an x in X ”.

Let G be a graph and let π be a path in G . By Lemma 1, there exist paths π_1 , π_2 and a closed path γ such that $\pi = \pi_1 \cdot \gamma \cdot \pi_2$. Because $\text{End}(\pi_1) = \text{Start}(\pi_2)$, the concatenation $\pi_1 \cdot \pi_2$ is well-defined. If π is non-simple, then we choose γ to be nonempty, i.e., $\ell(\pi_1 \cdot \pi_2) < \ell(\pi)$.

We define $\text{Step}(\pi)$ to be the concatenation $\pi_1 \cdot \pi_2$. If π is simple, then $\text{Step}(\pi) = \pi$. Furthermore, we define

$$\text{Simp}(\pi) = \lim_{t \rightarrow \infty} \text{Step}^t(\pi) .$$

The construction of $\text{Simp}(\pi)$ takes a finite number of (at most $\ell(\pi)$) steps, hence $\text{Simp}(\pi)$ is well-defined. Since $\text{Step}(\pi) = \pi$ if and only if π is simple, $\text{Simp}(\pi)$ is simple. Finally, π and $\text{Step}(\pi)$, and so π and $\text{Simp}(\pi)$, have the same start and end nodes, respectively. We call $\text{Simp}(\pi)$ the *simple part* of π .

3.2 The critical bound

Lemma 12. *Let G be a nontrivial strongly connected en -weighted graph and let π be a path in G whose nodes are non-critical. Then,*

$$w(\pi) \leq w(\text{Simp}(\pi)) + \varrho_{nc}(G) \cdot (\ell(\pi) - \ell(\text{Simp}(\pi))) .$$

Proof. It suffices to show the inequality with $\text{Step}(\pi)$ instead of $\text{Simp}(\pi)$.

If $\text{Step}(\pi) = \pi$, then the inequality trivially holds. Otherwise, let γ be the nonempty closed path in the definition of $\text{Step}(\pi)$. Then $w(\pi) = w(\text{Step}(\pi)) + w_*(\gamma)$. By assumption, γ is a nonempty closed path whose nodes are non-critical, hence $w_*(\gamma) \leq \varrho_{nc}(G) \cdot \ell(\gamma)$. Noting $\ell(\gamma) = \ell(\pi) - \ell(\text{Step}(\pi))$ concludes the proof. \square

We introduce some additional notation for a en -weighted graph G : Let $cd_{nc}(G)$ be the length of the longest simple path in G whose nodes are noncritical, and let $cr_c(G)$ be the length of the

¹We could also restrict the universe of possible nodes to a given set \mathcal{U} and explicitly state a choice function.

longest elementary critical closed path in G . We define

$$\|w_V\| = \max_{i \in V} w_V(i) - \min_{i \in V} w_V(i) .$$

Analogously to $\|w_V\|$, we define for vectors $v \in \mathbb{R}_{\max}^N$:

$$\|v\| = \max_{1 \leq i \leq N} v_i - \min_{1 \leq i \leq N} v_i$$

To enhance readability and since no confusion can arise, we omit the dependency on the graph G in the next definition:

$$B_c(G) = \min \left\{ cd_{nc} + \frac{\|w_V\| + \overline{\Delta}_{nc} cd_{nc} - \bar{\delta} cd}{-\bar{\varrho}_{nc}} , \frac{\|w_V\| + \overline{\Delta}_{nc} cd_{nc} - \bar{\delta} (N_{nc} + cr_c - 1)}{-\bar{\varrho}_{nc}} \right\} , \quad (4)$$

where N_{nc} is the number of non-critical nodes of G .

Theorem 2. *Let G be a nontrivial strongly connected en-weighted graph and let i be a node of G . For all $n \geq B_c(G)$, there exists a path of maximum en-weight in $\mathcal{P}^n(i \rightarrow, G)$ that contains a critical node.*

Proof. Since a path is of maximum en-weight in $\mathcal{P}^n(i \rightarrow, \overline{G})$ if and only if it is of maximum en-weight in $\mathcal{P}^n(i \rightarrow, G)$, the critical nodes in \overline{G} and G are the same, and $B_c(\overline{G}) = B_c(G)$, we may assume without loss of generality. that $\varrho(G) = 0$ in the following.

Let $\overline{\Delta}_{nc} = \overline{\Delta}_{nc}(G)$, $\bar{\delta} = \bar{\delta}(G)$, and $\bar{\varrho}_{nc} = \bar{\varrho}_{nc}(G)$.

Now suppose by contradiction that there exists an $n \geq B_c(G)$ such that all paths of maximum weight in $\mathcal{P}^n(i \rightarrow, G)$ are paths with non-critical nodes only. Let $\hat{\pi}$ be a path in $\mathcal{P}^n(i \rightarrow, G)$ of maximum weight with non-critical nodes only.

Next choose a critical node k and a prefix π_c of $\text{Simp}(\hat{\pi})$, such that the distance between k and $\text{End}(\pi_c)$ is minimal. Let π_2 be a path of minimum length from $\text{End}(\pi_c)$ to k . Further let γ be a critical elementary closed path with $\text{Start}(\gamma) = \text{End}(\gamma) = k$. Choose $m \in \mathbb{N}$ to be maximal such that $\ell(\pi_c) + \ell(\pi_2) + m \cdot \ell(\gamma) \leq n$ and choose π_1 to be a prefix of γ of length $n - (\ell(\pi_c) + \ell(\pi_2) + m \cdot \ell(\gamma))$. Clearly $\text{Start}(\pi_1) = k$. If we set $\pi = \pi_c \cdot \pi_2 \cdot \gamma^m \cdot \pi_1$, we get $\ell(\pi) = n$ and for the weight of π in G ,

$$w(\pi) \geq \min_{j \in V} w_V(j) + w_*(\pi_c) + w_*(\pi_2) + w_*(\pi_1) . \quad (5)$$

Figure 1 illustrates path π .

Let π_3 be a path such that $\text{Simp}(\hat{\pi}) = \pi_c \cdot \pi_3$. By Lemma 12 we obtain for the weight of $\hat{\pi}$ in G ,

$$\begin{aligned} w(\hat{\pi}) &\leq w(\text{Simp}(\hat{\pi})) + \bar{\varrho}_{nc} \cdot (\ell(\hat{\pi}) - \ell(\text{Simp}(\hat{\pi}))) \\ &\leq \max_{j \in V} w_V(j) + w_*(\pi_c) + w_*(\pi_3) + \bar{\varrho}_{nc} \cdot (\ell(\hat{\pi}) - \ell(\pi_c) - \ell(\pi_3)) \end{aligned} \quad (6)$$

By assumption $w(\hat{\pi}) > w(\pi)$, and from (5), (6), and $\bar{\varrho}_{nc} < 0$ we therefore obtain

$$\begin{aligned} \ell(\hat{\pi}) &< \frac{\|w_V\| + w_*(\pi_3) - w_*(\pi_1) - w_*(\pi_2)}{-\bar{\varrho}_{nc}} + \ell(\pi_3) + \ell(\pi_c) \\ &\leq \frac{\|w_V\| + \overline{\Delta}_{nc} \ell(\pi_3) - \bar{\delta} (\ell(\pi_1) + \ell(\pi_2))}{-\bar{\varrho}_{nc}} + \ell(\pi_3) + \ell(\pi_c) \end{aligned} \quad (7)$$

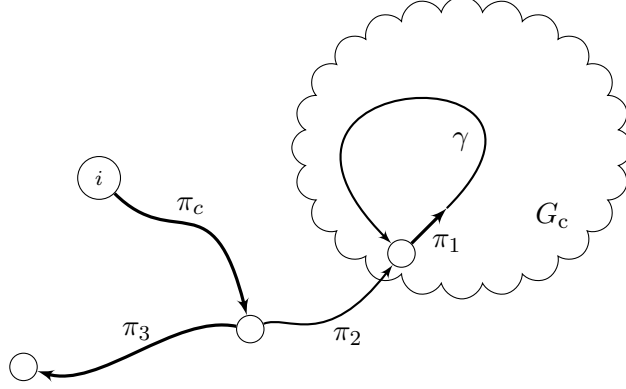


Figure 1: Path π in proof of Theorem 2

From (7) we may deduce,

$$\ell(\hat{\pi}) < \frac{\|w_V\| + \overline{\Delta}_{nc} cd_{nc}(G) - \overline{\delta} cd(G)}{-\overline{\varrho}_{nc}} + cd_{nc}(G) . \quad (8)$$

Alternatively we may deduce from (7) with $-\overline{\varrho}_{nc} \leq -\overline{\delta}$, $\ell(\pi_2) + \ell(\pi_3) + \ell(\pi_c) \leq N_{nc}$, and $\ell(\pi_1) \leq cr_c(G) - 1$ that

$$\ell(\hat{\pi}) < \frac{\|w_V\| + \overline{\Delta}_{nc} cd_{nc}(G) - \overline{\delta} (N_{nc} + cr_c(G) - 1)}{-\overline{\varrho}_{nc}} . \quad (9)$$

Combination of (8) and (9) yields a contradiction to $n \geq B_c(G)$. The lemma follows. \square

Even and Rajsbaum [4, Lemma 10] and Hartmann and Arguelles [9, Claim in proof of Theorem 10] arrived at analog bounds for critical nodes on maximum weight paths. A comparison of these bounds is given in Section 8.1

From Theorem 2 together with $cd_{nc}(G) \leq N - 1$ and $N_{nc} + cr_c(G) \leq N$, and Lemma 4 we immediately obtain:

Corollary 1. *Let G be a nontrivial strongly connected en-weighted graph with N nodes and let i be a node of G . For all*

$$n \geq \frac{\|w_V\| + (\Delta_{nc}(G) - \delta(G))(N - 1)}{\varrho(G) - \varrho_{nc}(G)} \geq B_c(G) ,$$

there exists a path of maximum en-weight in $\mathcal{P}^n(i \rightarrow, G)$ that contains a critical node.

4 Arbitrarily Long Closed Paths

We introduce for a strongly connected graph G the *exploration penalty* of G , $ep(G)$, as the smallest integer k such that for any node i and any integer $n \geq k$ that is a multiple of $c(G)$, there is a closed path of length n starting at i . We prove that $ep(G)$ is finite, and we give an upper bound on $ep(G)$ which is quadratic in the number of nodes of G . As we see in the subsequent sections, the exploration penalty plays a key role to bound the transient as it constitutes a threshold to “pump” path weights inside the critical graph.

4.1 A number-theoretic lemma

We now state a useful number-theoretic lemma, which is a simple application of Brauer's Theorem [12].

Let \mathbf{N} be any nonempty set of integers. Any nonempty subset $\mathbf{A} \subseteq \mathbf{N}$ is said to be a *gcd-generator* of \mathbf{N} if $\gcd(\mathbf{A}) = \gcd(\mathbf{N})$. Note that, as \mathbb{Z} is Noetherian, any nonempty set of integers admits a finite gcd-generator.

Lemma 13. *A set \mathbf{N} of positive integers that is closed under addition contains all but a finite number of multiples of its greatest common divisor. Moreover, if $\{a_1, \dots, a_k\}$ is a finite gcd-generator of \mathbf{N} with $a_1 \leq \dots \leq a_k$, then any multiple n of $d = \gcd(\mathbf{N})$ such that $n \geq d(\frac{a_1}{d} - 1)(\frac{a_k}{d} - 1)$ is in \mathbf{N} .*

Proof. Consider the set \mathbf{M} of all the elements in \mathbf{N} divided by $d = \gcd(\mathbf{N})$. By Brauer's Theorem [12], we know that every integer $m \geq (\frac{a_1}{d} - 1)(\frac{a_k}{d} - 1)$ is of the form

$$m = \sum_{i=1}^k x_i \frac{a_i}{d}$$

where each x_i is a nonnegative integer. Since \mathbf{N} is closed under addition, it follows that every multiple of d that is greater or equal to $d(\frac{a_1}{d} - 1)(\frac{a_k}{d} - 1)$ is in \mathbf{N} . In particular, all but a finite number of multiples of d are in \mathbf{N} . \square

4.2 Constructing long paths

In the case $G = (V, E)$ is a primitive graph, Denardo [13] established the following upper bound on $ep(G)$:

Lemma 14 (Denardo, [13, Corollary 1]). *Let G be a strongly connected primitive graph with N nodes and of girth g . For any integer $n \geq N + (N - 2)g$ and any node i of G , there exists a closed path starting at i of length n .*

In [14], we prove that the same upper bound actually holds for any (primitive or non-primitive) strongly connected graph. We now show that we can obtain a better upper bound on $ep(G)$ in the case of non-primitive graphs from the number-theoretic lemma in Section 4.1. For that, we first recall some well-known properties on the lengths of paths in a strongly connected graph. Let $\mathbf{N}_{i,j}$ be the subset of integers defined by:

$$\mathbf{N}_{i,j} = \{n \in \mathbb{N} \mid \exists \pi \in \mathcal{P}(i, j, G), n = \ell(\pi)\}.$$

Clearly each $\mathbf{N}_{i,i}$ is closed under addition; let $d_i = \gcd(\mathbf{N}_{i,i})$. Obviously,

$$c(G) = \gcd(\{d_i \mid i \in V\}). \tag{10}$$

Lemma 15. *For any node i in G , $d_i = c(G)$. Moreover, for any pair of nodes i, j , all the elements in $\mathbf{N}_{i,j}$ have the same residue modulo $c(G)$.*

Proof. Let i, j be any pair of nodes, and let $a \in \mathbf{N}_{i,j}$ and $b \in \mathbf{N}_{j,i}$. The concatenation of a path from i to j with a path from j to i is a closed path starting at i . Hence $a+b \in \mathbf{N}_{i,i}$. From Lemma 13, we know that $\mathbf{N}_{j,j}$ contains all the multiples of d_j greater than some integer. Consider any such multiple kd_j with k and d_i relatively prime integers. By inserting one corresponding closed path at node j into the closed path at i with length $a+b$, we obtain a new closed path starting at i , i.e., $a+kd_j+b \in \mathbf{N}_{i,i}$. It follows that d_i divides both $a+b$ and $a+kd_j+b$, and so d_i divides d_j . Similarly, we prove that d_j divides d_i , and so $d_i = d_j$. By (10), the common value of the d_i 's is actually equal to $c(G)$.

Let a and a' be two integers in $\mathbf{N}_{i,j}$. The above argument gives both $a+b$ and $a'+b$ in $\mathbf{N}_{i,i}$. Hence $c(G)$ divides $a+b$ and $a'+b$, and so $a-a'$. \square

Lemma 16. *For any node i of G , the set $\mathbf{N}_{i,i}$ admits a gcd-generator which contains all the lengths of elementary closed paths starting at i , and whose all elements n satisfy the inequality*

$$g \leq n \leq 2N - 1$$

where g is the girth of G and N is the number of nodes in G .

Proof. Let i be any node of G , and let γ_0 be any elementary closed path. Let π_1 be one of the shortest paths from i to γ_0 , and let $j = \text{End}(\pi_1)$. Without loss of generality, $\text{Start}(\gamma_0) = j$. By definition, $\ell(\pi_1) \leq N - \ell(\gamma_0)$. Then consider a simple path π_2 from j to i , and the two closed paths

$$\pi = \pi_1 \cdot \pi_2 \text{ and } \pi' = \pi_1 \cdot \gamma_0 \cdot \pi_2 .$$

Note that

$$\ell(\pi) \leq \ell(\pi') \leq 2N - 1$$

and $\ell(\pi) \geq g$, because π is closed. In the particular case i is a node of γ_0 , π' reduces to γ_0 , and so $\ell(\pi')$ is the length of the elementary closed path γ_0 .

Let \mathbf{N}_i be the set of the lengths of the closed paths π and π' when considering all the elementary closed paths γ_0 in G . Then, \mathbf{N}_i contains all the length of elementary closed paths starting at i . Let $g_i = \gcd(\mathbf{N}_i)$. Since $\mathbf{N}_i \subseteq \mathbf{N}_{i,i}$, d_i divides g_i . Conversely, let γ_0 be any elementary closed path, and let π and π' be the two closed paths starting at node i defined above; g_i divides both $\ell(\pi)$ and $\ell(\pi')$, and so divides $\ell(\pi') - \ell(\pi) = \ell(\gamma_0)$. Hence, g_i divides the length of any elementary closed path, i.e., g_i divides $c(G)$. By Lemma 15, it follows that g_i divides d_i . Consequently, $g_i = d_i$, that is to say \mathbf{N}_i is a gcd-generator of $\mathbf{N}_{i,i}$. \square

Lemma 17. *Let G be a strongly connected graph with N nodes, of girth g and cyclicity c . For any node i of G and any integer n such that n is a multiple of c and $n \geq 2Ng/c - g/c - 2g + c$, there exists a closed path of length n starting at i .*

Proof. Let i be any node, and let γ_0 be any elementary closed path such that $\ell(\gamma_0) = g$. Let π_1 be one of the shortest paths from i to γ_0 , and let $j = \text{End}(\pi_1)$. Without loss of generality, $\text{Start}(\gamma_0) = j$. By definition, $\ell(\pi_1) \leq N - g$. Then consider an elementary path π_2 from j to i ; we have $\ell(\pi_2) \leq N - 1$. The path $\pi_1 \cdot \pi_2$ is closed at node i , and so c divides $\ell(\pi_1) + \ell(\pi_2)$. Hence, if c divides some integer n , then c also divides $n - \ell(\pi_1) - \ell(\pi_2)$. It is $g \in \mathbf{N}_{j,j}$. By Lemma 16, there exists a gcd-generator \mathbf{N}_j of $\mathbf{N}_{j,j}$ such that $g \in \mathbf{N}_j$ and $g \leq n \leq 2N - 1$ for all $n \in \mathbf{N}_j$.

By Lemma 13, for any n such that $n' = n - \ell(\pi_1) - \ell(\pi_2)$ is a multiple of c and

$$n' \geq c \left(\frac{g}{c} - 1 \right) \left(\frac{2N-1}{c} - 1 \right) ,$$

there exists a closed path γ starting at node j of length $\ell(\gamma) = n'$. Note that

$$c \left(\frac{g}{c} - 1 \right) \left(\frac{2N-1}{c} - 1 \right) + (N - g) + (N - 1) = 2\frac{g}{c}N - \frac{g}{c} - 2g + c .$$

In this way, for any integer $n \geq 2Ng/c - g/c - 2g + c$ that is a multiple of c , we construct $\pi = \pi_1 \cdot \gamma \cdot \pi_2$ that is a closed path at node i of length n . \square

We easily check that the upper bound on $ep(G)$ given by Lemma 17 is better than the generalized Denardo bound [14], except when $c = 1$, since then $2 \leq c \leq g$ holds in the theorem below.

Theorem 3. *Let G be a strongly connected graph with N nodes, of girth g and cyclicity c . The exploration penalty of G , denoted ep , is well-defined and satisfies the inequality*

$$ep \leq 2\frac{g}{c}N - \frac{g}{c} - 2g + c$$

which can be improved to

$$ep \leq N + (N - 2)g$$

in case G is a primitive graph.

5 Explorative Bound

In this section, we show a bound on the transient $n_{A,v}$ of linear max-plus systems for irreducible matrices A and vectors v with only finite entries, i.e., $v \in \mathbb{R}^N$. We construct arbitrarily long paths by *exploring* a critical component H in the sense of Section 4; we hence call the resulting bound the *explorative bound*. As we did in Section 4, we distinguish the cases of whether the critical subgraph is primitive or not: If $c(H) = 1$, we can find critical closed paths in H of arbitrary length $t \geq ep(H)$; otherwise, it is only possible to find critical closed paths of length t if $c(H)$ divides t . We start with the case of a primitive critical subgraph in Section 5.2, because it allows for a simpler proof, as we do not have to consider the constraint that critical closed path lengths in critical component are necessarily multiples of $c(H)$.

5.1 Outline of the proof

We now show how to establish an upper bound $B(A, v)$ on the transient of the sequence $(A^{\otimes n} \otimes v)_i$ having the properties that (i) $B(A, v)$ is greater or equal to the critical bound $B_c(G(A, v))$ and (ii) $B(A, v)$ is invariant under substituting A by \bar{A} . By Lemmata 3 and 10, and property (ii), it suffices to establish the upper bound in the case $\varrho(A) = 0$. Let $B = B(A, v)$.

First, we take any $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer π for node i in en-weighted graph $G = G(A, v)$; realizer π exists by Lemma 11. By property (i) and Theorem 2, we know that we can choose π to contain a critical node k . Depending on k , we choose a modulus d dividing $p(G)$. In Section 5.3, we introduce a path reduction—a generalization of the simple part of a path—that preserves the residue class of

the reduced path, i.e., $\ell(\pi) \equiv \ell(\hat{\pi}) \pmod{d}$ where $\hat{\pi}$ is the reduced path of π . We apply our path reduction $\text{Red}_{d,k}$ to π , obtaining path $\hat{\pi}$. Lemma 20 in Section 5.3 shows that the length of $\hat{\pi}$ is less or equal to some bound $B_{\text{Red}}(d)$. Further, k is a node of $\hat{\pi}$ and $w(\hat{\pi}) \geq w(\pi)$.

We then show that, for arbitrary multiples t of d such that $t \geq B - B_{\text{Red}}(d)$, there exist critical closed paths starting at k of length t . It follows that for all $n \geq B$, there exists an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer π_n for i of length n . Application of Lemma 2 then concludes the proof.

5.2 The case of primitive critical subgraphs

For a nontrivial strongly connected en-weighted graph G with primitive critical subgraph, define

$$B_e(G) = \max \left\{ B_c(G), 2 \cdot cd(G) + \max_{H \in \mathcal{C}(G_c)} ep(H) \right\}.$$

Lemma 18. *Let G be a nontrivial strongly connected en-weighted graph with primitive critical subgraph and $\varrho(G) = 0$; let $B = B_e(G)$ and $p = p(G)$. For all nodes i and all $n \geq B$ exists an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer for i of length n .*

Proof. By Lemma 11, there exists an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer π for i . By Theorem 2, we can choose π to contain a critical node k , because $B \geq B_c(G)$. Let $\pi = \pi_1 \cdot \pi_2$ with $\text{End}(\pi_1) = k$.

Set $\hat{\pi}_1 = \text{Simp}(\pi_1)$ and $\hat{\pi}_2 = \text{Simp}(\pi_2)$. Because $\varrho(G) = 0$, it is $\varrho_{\text{nc}}(G) \leq 0$, and hence $w_*(\hat{\pi}_1) \geq w_*(\pi_1)$ and $w(\hat{\pi}_2) \geq w(\pi_2)$ by Lemma 12. The lengths of $\hat{\pi}_1$ and $\hat{\pi}_2$ satisfy $\ell(\hat{\pi}_1) \leq cd(G)$ and $\ell(\hat{\pi}_2) \leq cd(G)$, because the paths are simple.

Let $H \in \mathcal{C}(G_c)$ be the strongly connected component of G_c in which critical node k is contained. Note that H is primitive, i.e., $c(H) = 1$. Thus, for all $t \geq ep(H)$ there exists a critical closed path γ_t of length t starting at node k .

By Lemma 7, $w_*(\gamma_t) = \varrho(G) = 0$, and hence

$$w(\hat{\pi}_1 \cdot \gamma_t \cdot \hat{\pi}_2) = w_*(\hat{\pi}_1) + w_*(\gamma_t) + w(\hat{\pi}_2) \geq w_*(\pi_1) + w(\pi_2) = w(\pi). \quad (11)$$

Further, $\ell(\hat{\pi}_1 \cdot \gamma_t \cdot \hat{\pi}_2) = \ell(\hat{\pi}_1) + \ell(\hat{\pi}_2) + t$.

We set $t = n - (\ell(\hat{\pi}_1) + \ell(\hat{\pi}_2)) \geq B - 2 \cdot cd(G) \geq ep(H)$ and $\pi_n = \hat{\pi}_1 \cdot \gamma_t \cdot \hat{\pi}_2$. Figure 2 depicts path π_n . It follows that $\ell(\pi_n) = n$ and $w(\pi_n) \geq w(\pi)$. Hence, because $n \geq B$ and $\text{Start}(\pi_n) = \text{Start}(\pi) = i$, π_n is an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer for i of length n . \square

Theorem 4. *For all irreducible $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ such that $G_c(A)$ is primitive, and all $v \in \mathbb{R}^N$, we have $n_{A,v} \leq B_e(G(A, v))$.*

Proof. By Lemma 10 and Lemma 4, both $n_{A,v}$ and $B_e(G(A, v))$ are invariant under substituting A by \bar{A} . We may hence assume without loss of generality that $\varrho(A) = 0$, i.e., $\varrho(G(A, v)) = 0$.

Let $B = B_e(G(A, v))$. Graph $G(A, v)$ is nontrivial and strongly connected, because A is irreducible. By Lemma 18, for every node i and every $n \in \mathbf{N}_{\geq B}^{(n,p)}$, there exists an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer for i of length n . Lemma 2 hence implies that the transient of sequence $(w^n(i \rightarrow))_n$ in graph $G(A, v)$ is at most B . But, by Lemma 8, the transient of $(w^n(i \rightarrow))_n$ in $G(A, v)$ is equal to $n_{A,v}$. This concludes the proof. \square

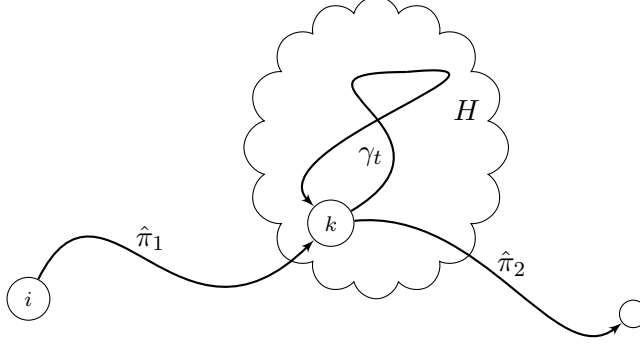


Figure 2: Realizer π_n in proof of Lemma 18

Corollary 2. *For all irreducible $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ such that $G_c(A)$ is primitive, and all $v \in \mathbb{R}^N$, we have*

$$n_{A,v} \leq \max \left\{ \frac{\|v\| + (\Delta_{\text{nc}}(G) - \delta(G)) \cdot (N - 1)}{\varrho(G) - \varrho_{\text{nc}}(G)}, 2 \cdot (N - 1) + \max_{H \in \mathcal{C}(G_c)} ep(H) \right\},$$

where $G = G(A)$.

5.3 Our second path reduction

In the case that the critical subgraph is not primitive, i.e., if there exist critical components H with $c(H) > 1$, our first path reduction—the simple part of a path—is not sufficient for us, because we can no longer choose a critical path γ_t in H of arbitrary length $t \geq ep(H)$, but only of lengths that are multiples of $c(H)$. In this section, we thus introduce a generalization of our first path reduction $\text{Simp}(\pi)$, which is able to preserve the residue class modulo $c(H)$ of the path lengths. The idea is to repeatedly remove *collections* of closed subpaths of π whose combined length is a multiple of $c(H)$.

The generalized path reduction, $\text{Red}_{d,k}(\pi)$, has two additional parameters: a modulus d and a node k of π . The reduction is defined in such a way that (a) the reduced path's length is in the same residue class modulo d as path π 's length, and (b) node k is a node of the reduced path. Our first path reduction is a special case of this second path reduction when setting $d = 1$ and k to be the start node of π .

Let G be a graph and let π be a path in G . Let \mathcal{S} be a finite multiset of subpaths of π . We say that \mathcal{S} is *disjoint* if there exist paths $\sigma_0, \sigma_1, \dots, \sigma_n$ such that

$$\pi = \sigma_0 \cdot \pi_1 \cdot \sigma_1 \cdots \pi_n \cdot \sigma_n \tag{12}$$

where $\mathcal{S} = \{\pi_1, \pi_2, \dots, \pi_n\}$.

For a disjoint multiset \mathcal{S} of *closed* subpaths of π , define

$$\text{Rem}(\pi, \mathcal{S}) = \sigma_0 \cdot \sigma_1 \cdots \sigma_n$$

where the σ_l are chosen to fulfill Equation (12). The paths π and $\text{Rem}(\pi, \mathcal{S})$ have the same start and end nodes, respectively. Furthermore $\ell(\text{Rem}(\pi, \mathcal{S})) = \ell(\pi) - L(\mathcal{S})$ where $L(\mathcal{S}) = \sum_{\gamma \in \mathcal{S}} \ell(\gamma)$.

In particular, $\text{Rem}(\pi, \mathcal{S}) = \pi$ if and only if $L(\mathcal{S}) = 0$. Hence if $\text{Rem}(\pi, \mathcal{S}) \neq \pi$, then necessarily $\ell(\text{Rem}(\pi, \mathcal{S})) < \ell(\pi)$.

For a path π and a node k of π , denote by $\mathbf{S}_k(\pi)$ the set of disjoint multisets \mathcal{S} of elementary closed subpaths of π such that k is a node of $\text{Rem}(\pi, \mathcal{S})$. For a path π , a node k of π , and a positive integer d , define

$$\mathbf{S}_{d,k}(\pi) = \{\mathcal{S} \in \mathbf{S}_k(\pi) \mid L(\mathcal{S}) \equiv 0 \pmod{d}\} .$$

This set is never empty, because k is a node of π and we can hence choose \mathcal{S} to be empty.

Choose $\mathcal{S} \in \mathbf{S}_{d,k}(\pi)$ such that $L(\mathcal{S})$ is maximal. Then set $\text{Step}_{d,k}(\pi) = \text{Rem}(\pi, \mathcal{S})$ and

$$\text{Red}_{d,k}(\pi) = \lim_{t \rightarrow \infty} \text{Step}_{d,k}^t(\pi) .$$

The construction of $\text{Red}_{d,k}(\pi)$ takes a finite number of (at most $\ell(\pi)$) steps, hence $\text{Red}_{d,k}(\pi)$ is well-defined. It is $\text{Red}_{d,k}(\pi) = \pi$ if and only if $L(\mathcal{S}) = 0$ for all $\mathcal{S} \in \mathbf{S}_{d,k}(\pi)$. The paths π and $\text{Red}_{d,k}(\pi)$ have the same start and end nodes, respectively. Also, k is a node of $\text{Red}_{d,k}(\pi)$ and

$$\ell(\text{Red}_{d,k}(\pi)) \equiv \ell(\pi) \pmod{d} .$$

Finally, whenever G is an en-weighted graph with $\varrho(G) = 0$, $w(\text{Rem}(\pi, \mathcal{S})) \geq w(\pi)$, because we repeatedly remove closed subpaths.

The following lemma is a well-known elementary application of the pigeonhole principle. Erdős attributed it to Vázsonyi and Sved. [15, p. 133]

Lemma 19. *Let d be a positive integer and let $x_1, \dots, x_d \in \mathbb{Z}$. Then there exists a nonempty $I \subseteq \{1, \dots, d\}$ such that*

$$\sum_{i \in I} x_i \equiv 0 \pmod{d} .$$

Proof. For $1 \leq k \leq d$, set $S_k = \sum_{i=1}^k x_i$. If there exist $k < \ell$ with $S_k \equiv S_\ell \pmod{d}$, then set $I = \{k+1, k+2, \dots, \ell\}$. Otherwise the mapping $k \mapsto (S_k \bmod d)$ is injective, hence surjective onto $\{0, 1, \dots, d-1\}$, which implies that there exists a k_0 with $S_{k_0} \equiv 0 \pmod{d}$. In this case, set $I = \{1, 2, \dots, k_0\}$. \square

With the help of Lemma 19, we can prove the following upper bound on the length of the reduced path $\text{Red}_{d,k}(\pi)$.

Lemma 20. *Let G be a graph with N nodes. For all positive integers d , all nodes k , and all paths π that contain node k ,*

$$\ell(\text{Red}_{d,k}(\pi)) \leq (d-1) \cdot cr(G) + (d+1) \cdot cd(G) .$$

Proof. Write $\hat{\pi} = \text{Red}_{d,k}(\pi)$. Let $\hat{\mathcal{S}} \in \mathbf{S}_k(\hat{\pi})$ such that $L(\hat{\mathcal{S}})$ is maximal. Further, let \mathcal{S} be the sub-multiset of the *nonempty* closed paths of $\hat{\mathcal{S}}$. It is $L(\mathcal{S}) = L(\hat{\mathcal{S}})$. Because $\hat{\mathcal{S}} \in \mathbf{S}_k(\hat{\pi})$ and $\mathcal{S} \subseteq \hat{\mathcal{S}}$, it follows that $\mathcal{S} \in \mathbf{S}_k(\hat{\pi})$.

We first show that $|\mathcal{S}| \leq d-1$: Otherwise, by Lemma 19, there exists a nonempty $\mathcal{S}' \subseteq \mathcal{S}$ with $L(\mathcal{S}') = \sum_{\gamma \in \mathcal{S}'} \ell(\gamma) \equiv 0 \pmod{d}$; hence $\mathcal{S}' \in \mathbf{S}_{d,k}(\hat{\pi})$ with $L(\mathcal{S}') > 0$, a contradiction to $\text{Red}_{d,k}(\hat{\pi}) = \hat{\pi}$. We have thus proved

$$|\mathcal{S}| \leq d-1 . \tag{13}$$

Every $\gamma \in \mathcal{S}$ is elementary, therefore $\ell(\gamma) \leq cr(G)$ and thus

$$L(\mathcal{S}) \leq (d-1) \cdot cr(G) . \quad (14)$$

Let $n = |\mathcal{S}|$, $\mathcal{S} = \{\gamma_1, \dots, \gamma_n\}$, and

$$\hat{\pi} = \sigma_0 \cdot \gamma_1 \cdot \sigma_1 \cdots \gamma_n \cdot \sigma_n .$$

Because k is a node of $\text{Rem}(\hat{\pi}, \mathcal{S})$, there exists an r such that k is a node of σ_r . Figure 3 shows this decomposition of path $\hat{\pi}$.

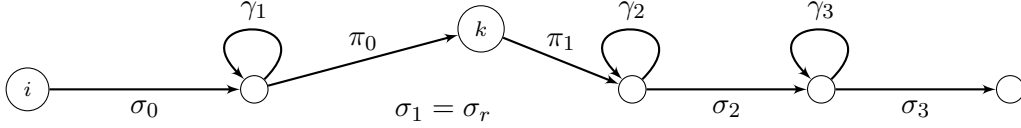


Figure 3: Path $\hat{\pi}$ in proof of Lemma 20

If $m \neq r$, then we show that σ_m is simple: Otherwise by Lemma 1, there exists a nonempty elementary closed subpath γ' of σ_m . But then $\mathcal{S}' = \mathcal{S} \cup \{\gamma'\} \in \mathbf{S}_k(\hat{\pi})$, because k is a node of σ_r and $m \neq r$; a contradiction to maximality of $L(\mathcal{S})$, because $L(\mathcal{S}') > L(\mathcal{S})$. Hence

$$\ell(\sigma_m) \leq cd(G) \text{ for } m \neq r . \quad (15)$$

We now show that $\ell(\sigma_r) \leq 2 \cdot cd(G)$: Let $\sigma_r = \pi_0 \cdot \pi_1$ with $\text{End}(\pi_0) = k$. Then π_0 and π_1 are simple, because otherwise, if γ'_t is a nonempty elementary closed subpath of π_t , $t \in \{0, 1\}$, then $\mathcal{S}' = \mathcal{S} \cup \{\gamma'_t\} \in \mathbf{S}_k(\hat{\pi})$, because k is a node of π_{1-t} ; a contradiction to the maximality of $L(\mathcal{S})$, because $L(\mathcal{S}') > L(\mathcal{S})$. We have thus proved that

$$\ell(\sigma_r) \leq 2 \cdot cd(G) . \quad (16)$$

We note that $n = |\mathcal{S}| \leq d-1$ by (13). Thus combination of (15) and (16) yields

$$\sum_{m=0}^n \ell(\sigma_m) \leq (d+1) \cdot cd(G) , \quad (17)$$

which, in turn, yields by combination with (14):

$$\ell(\hat{\pi}) = L(\mathcal{S}) + \sum_{m=0}^n \ell(\sigma_m) \leq (d-1) \cdot cr(G) + (d+1) \cdot cd(G)$$

This concludes the proof. \square

5.4 Extension to the general case

In this section, we generalize the explorative bound to the case of a not necessarily primitive critical subgraph. For that, we use the generalized path reduction $\text{Red}_{d,k}(\pi)$.

For a nontrivial strongly connected en-weighted graph G define

$$B_e(G) = \max \left\{ B_c(G) , (d(G_c) - 1) \cdot cr(G) + (d(G_c) + 1) \cdot cd(G) + \max_{H \in \mathcal{C}(G_c)} ep(H) \right\} .$$

This definition generalizes the previous definition of $B_e(G)$, which was stated for the case of a primitive critical subgraph.

Lemma 21. *Let G be a nontrivial strongly connected en-weighted graph with $\varrho(G) = 0$; let $B = B_e(G)$ and $p = p(G)$. For all nodes i and all $n \geq B$ exists an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer for i of length n .*

Proof. By Lemma 11, there exists an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer π for i , i.e., $\pi \in P(i \rightarrow)$ is of maximum weight such that $\ell(\pi) \geq B$ and $\ell(\pi) \equiv n \pmod{p}$. By Theorem 2, we can choose π such that there is a critical node k on π , because $B \geq B_c(G)$. Let $H \in \mathcal{C}(G_c)$ be the critical component of k .

Set $\hat{\pi} = \text{Red}_{c(H),k}(\pi)$. It is $\ell(\hat{\pi}) \equiv \ell(\pi) \pmod{c(H)}$ and also $\ell(\pi) \equiv n \pmod{c(H)}$, because $c(H)$ divides $p(G)$. Hence $\ell(\hat{\pi}) \equiv n \pmod{c(H)}$, i.e., there exists an integer m such that $n = \ell(\hat{\pi}) + m \cdot c(H)$. It follows from Lemma 20 and $c(H) \leq d(G_c)$ that $m \cdot c(H) = n - \ell(\hat{\pi}) \geq ep(H)$; in particular, m is nonnegative. Hence, by Lemma 17, there exists a closed path γ_t of length $t = m \cdot c(H)$ in H starting at node k . By Lemma 7, $w_*(\gamma_t) = \varrho(G) = 0$.

Let $\hat{\pi} = \hat{\pi}_1 \cdot \hat{\pi}_2$ with $\text{End}(\hat{\pi}_1) = k$ and set $\pi_n = \hat{\pi}_1 \cdot \gamma_t \cdot \hat{\pi}_2$. Path π_n has the same form as in Figure 2. Then $\ell(\pi_n) = \ell(\hat{\pi}) + \ell(\gamma_t) = n$ and

$$w(\pi_n) = w_*(\hat{\pi}_1) + w_*(\gamma_t) + w(\hat{\pi}_2) = w(\hat{\pi}_1 \cdot \hat{\pi}_2) = w(\hat{\pi}) \geq w(\pi) .$$

Hence, because $\text{Start}(\pi_n) = \text{Start}(\pi) = i$, π_n is an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer for i of length n . \square

Theorem 5. *For all irreducible $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ and all $v \in \mathbb{R}^N$, we have $n_{A,v} \leq B_e(G(A, v))$.*

Proof. By Lemma 10 and Lemma 4, both $n_{A,v}$ and $B_e(G(A, v))$ are invariant under substituting A by \bar{A} . We may hence assume without loss of generality that $\varrho(A) = 0$, i.e., $\varrho(G(A, v)) = 0$.

Let $B = B_e(G(A, v))$ and $p = p(G)$. Graph $G(A, v)$ is nontrivial and strongly connected, because A is irreducible. By Lemma 21, for every node i and every $n \geq B$, there exists an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer for i of length n . Lemma 2 hence implies that the transient of sequence $(w^n(i \rightarrow))_n$ in graph $G(A, v)$ is at most B . But, by Lemma 8, the transient of $(w^n(i \rightarrow))_n$ in $G(A, v)$ is equal to $n_{A,v}$. This concludes the proof. \square

Corollary 3. *For all irreducible $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ and all $v \in \mathbb{R}^N$, we have*

$$n_{A,v} \leq \max \left\{ \frac{\|v\| + (\Delta_{\text{nc}}(G) - \delta(G)) \cdot (N - 1)}{\varrho(G) - \varrho_{\text{nc}}(G)} , (d - 1) + 2d \cdot (N - 1) + \max_{H \in \mathcal{C}(G_c)} ep(H) \right\} ,$$

where $G = G(A)$ and $d = d(G_c)$.

6 Repetitive Bounds

In Section 5, we explored critical components H in the sense of Section 4 to construct critical closed paths whose lengths are multiples of $c(H)$. We now follow another approach, which yields better results in some cases: We do not explore anymore critical components, but instead consider a single (elementary) critical closed path γ starting at critical node k , and repeatedly add γ to the constructed paths; we hence call the resulting bounds *repetitive*. We present two repetitive bounds: one using our path reduction $\text{Red}_{d,k}$ in Section 6.1, and an improvement of the bound of Hartmann and Arguëlles [9] by a factor of two in Section 6.2.

6.1 Our repetitive bound

This section presents the bound we get when substituting the exploration of the visited critical component by only using a single elementary critical closed path in the method used to derive the explorative bound. The proof follows the same outline as described in Section 5.1. In the resulting upper bound on the transient, we substitute $c(H)$ by $\ell(\gamma)$, and thus substitute $d(G_c)$ by $cr(G_c)$, and drop the exploration penalty terms $ep(H)$. It is of course possible that $cr(G_c)$ is strictly greater than $d(G_c)$.

For a nontrivial strongly connected en-weighted graph G define

$$B_r(G) = \max \left\{ B_c(G), (cr(G_c) - 1) \cdot cr(G) + (cr(G_c) + 1) \cdot cd(G) \right\}.$$

Lemma 22. *Let G be a nontrivial strongly connected en-weighted graph with $\varrho(G) = 0$; let $B = B_r(G)$ and $p = p(G)$. For all nodes i and all $n \geq B$ exists an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer for i of length n .*

Proof. By Lemma 11, there exists an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer π for i , i.e., $\pi \in P(i \rightarrow)$ is of maximum weight such that $\ell(\pi) \geq B$ and $\ell(\pi) \equiv n \pmod{p}$. By Theorem 2, we can choose π such that there is a critical node k on π , because $B \geq B_c(G)$. Let $\gamma \in \mathcal{C}(G_c)$ be an elementary critical closed path starting at k .

Set $\hat{\pi} = \text{Red}_{\ell(\gamma), k}(\pi)$. It is $\ell(\hat{\pi}) \equiv \ell(\pi) \pmod{\ell(\gamma)}$ and also $\ell(\pi) \equiv n \pmod{\ell(\gamma)}$, because $\ell(\gamma)$ divides $p(G)$. Hence $\ell(\hat{\pi}) \equiv n \pmod{\ell(\gamma)}$, i.e., there exists an integer m such that $n = \ell(\hat{\pi}) + m \cdot \ell(\gamma)$. Lemma 20 and $\ell(\gamma) \leq cr(G_c)$ implies $\ell(\hat{\pi}) \leq B$, hence m is nonnegative.

Let $\hat{\pi} = \hat{\pi}_1 \cdot \hat{\pi}_2$ with $\text{End}(\hat{\pi}_1) = k$ and set $\pi_n = \hat{\pi}_1 \cdot \gamma^m \cdot \hat{\pi}_2$. Figure 4 depicts path π_n . Then $\ell(\pi_n) = \ell(\hat{\pi}) + m \cdot \ell(\gamma) = n$. Further, $w(\pi_n) = w(\hat{\pi}) \geq w(\pi)$ and $\text{Start}(\pi_n) = \text{Start}(\pi) = i$, i.e., π_n is an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer for i if length n . \square

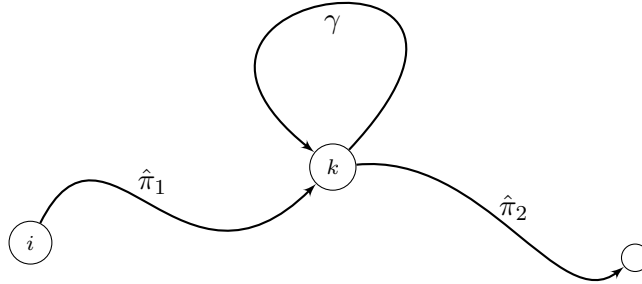


Figure 4: Realizer π_n in proof of Lemma 22

Theorem 6. *For all irreducible $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ and all $v \in \mathbb{R}^N$, we have $n_{A,v} \leq B_r(G(A, v))$.*

Proof. By Lemma 10 and Lemma 4, both $n_{A,v}$ and $B_r(G(A, v))$ are invariant under substituting A by \bar{A} . We may hence assume without loss of generality that $\varrho(A) = 0$, i.e., $\varrho(G(A, v)) = 0$.

Let $B = B_r(G(A, v))$ and $p = p(G)$. Graph $G(A, v)$ is nontrivial and strongly connected, because A is irreducible. By Lemma 22, for every node i and every $n \geq B$, there exists an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer for i of length n . Lemma 2 hence implies that the transient of sequence $(w^n(i \rightarrow))_n$ in graph $G(A, v)$ is at most B . But, by Lemma 8, the transient of $(w^n(i \rightarrow))_n$ in $G(A, v)$ is equal to $n_{A,v}$. This concludes the proof. \square

Corollary 4. *For all irreducible $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ and all $v \in \mathbb{R}^N$, we have*

$$n_{A,v} \leq \max \left\{ \frac{\|v\| + (\Delta_{\text{nc}}(G) - \delta(G)) \cdot (N - 1)}{\varrho(G) - \varrho_{\text{nc}}(G)}, (cr - 1) + 2cr \cdot (N - 1) \right\}$$

where $G = G(A)$ and $cr = cr(G_c)$.

6.2 Improved Hartmann-Arguelles bound

In Corollary 4, the second term in the maximum is bounded by $2N^2$. Hartmann and Arguelles also arrived at the term $2N^2$ in the corresponding part of their upper bound on the transient of a max-plus system [9, Theorem 12]. To prove this upper bound, Hartmann and Arguelles used a different kind of path reduction, which we recall in Theorem 7 below. This path reduction guarantees that the reduced path is in the same residue class modulo $\ell(\gamma)$, the length of some visited critical closed path γ .

Hartmann and Arguelles, after reducing a realizer with their path reduction, combine elementary critical closed paths in the visited critical component to arrive at the right residue class modulo $c(G_c)$. However, this construction is not necessary, for it is not necessary to consider $c(G_c)$ as the period, as we have shown in Section 2.5. We can improve on their proof by considering $p(G)$ as the period, which is a multiple of $c(G_c)$. Avoiding the construction saves one time N^2 in the resulting bound; hence the second term in the maximum can be chosen to be N^2 instead of $2N^2$, as we show now.

Theorem 7 ([9, Theorem 4]). *Let G be an e -weighted graph with N nodes and $\varrho(G) = 0$. Let $\pi \in \mathcal{P}(i, j, G)$ with $\ell(\pi) \geq N^2$ and let k be a critical node of π . Further, let γ be a critical closed path starting from k . Then there exist a path $\hat{\pi} \in \mathcal{P}(i, j, G)$ such that $\ell(\hat{\pi}) \equiv \ell(\pi) \pmod{\ell(\gamma)}$, $w_*(\hat{\pi}) \geq w_*(\pi)$, k is a node of $\hat{\pi}$, and $\ell(\hat{\pi}) < N^2$. \square*

For a nontrivial strongly connected en-weighted graph G with N nodes define

$$B_r^{\text{HA}}(G) = \max \{ B_c(G), N^2 \}.$$

Lemma 23. *Let G be a nontrivial strongly connected en-weighted graph with $\varrho(G) = 0$; let $B = B_r^{\text{HA}}(G)$ and $p = p(G)$. For all nodes i and all $n \geq B$ exists an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer for i of length n .*

Proof. By Lemma 11, there exists a $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer π for i , i.e., $\pi \in \mathcal{P}(i \rightarrow)$ is of maximum weight such that $\ell(\pi) \geq B$ and $\ell(\pi) \equiv n \pmod{p}$. By Theorem 2, we can choose π such that there is a critical node k on π , because $B \geq B_c$. Let γ be an elementary critical closed path starting at k .

Let $\hat{\pi}$ be as in Theorem 7 and let $\hat{\pi} = \hat{\pi}_1 \cdot \hat{\pi}_2$ with $\text{End}(\hat{\pi}_1) = k$. It is $\ell(\hat{\pi}) \equiv \ell(\pi) \pmod{\ell(\gamma)}$ and also $\ell(\pi) \equiv r \pmod{\ell(\gamma)}$, because $\ell(\gamma)$ divides $p(G)$. Hence $\ell(\hat{\pi}) \equiv n \pmod{\ell(\gamma)}$, i.e., there exists an integer m such that $t = m \cdot \ell(\gamma) = n - \ell(\hat{\pi})$. It is $\ell(\hat{\pi}) < N^2 \leq B$, hence m is nonnegative.

Setting $\pi_n = \hat{\pi}_1 \cdot \gamma^m \cdot \hat{\pi}_2$ yields $\ell(\pi_n) = n$ and $w(\pi_n) = w(\hat{\pi}) \geq w(\pi)$. This concludes the proof. \square

Theorem 8. *For all irreducible $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ and all $v \in \mathbb{R}^N$, we have $n_{A,v} \leq B_r^{\text{HA}}(G)$.*

Proof. By Lemma 10 and Lemma 4, both $n_{A,v}$ and $B_r^{\text{HA}}(G(A, v))$ are invariant under substituting A by \bar{A} . We may hence assume without loss of generality that $\varrho(A) = 0$, i.e., $\varrho(G(A, v)) = 0$.

Let $B = B_r^{\text{HA}}(G(A, v))$ and $p = p(G)$. Graph $G(A, v)$ is nontrivial and strongly connected, because A is irreducible. By Lemma 23, for every node i and every $n \geq B$, there exists an $\mathbf{N}_{\geq B}^{(n,p)}$ -realizer for i of length n . Lemma 2 hence implies that the transient of sequence $(w^n(i \rightarrow))_n$ in graph $G(A, v)$ is at most B . But, by Lemma 8, the transient of $(w^n(i \rightarrow))_n$ in $G(A, v)$ is equal to $n_{A,v}$. This concludes the proof. \square

Corollary 5. *For all irreducible $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ and all $v \in \mathbb{R}^N$, we have*

$$n_{A,v} \leq \max \left\{ \frac{\|v\| + (\Delta_{\text{nc}}(G) - \delta(G)) \cdot (N - 1)}{\varrho(G) - \varrho_{\text{nc}}(G)}, N^2 \right\},$$

where $G = G(A)$.

7 Matrix vs. System Transients

To obtain an upper bound on the transient of a max-plus matrix, we can follow the same idea as in Sections 3, 5, and 6, and substitute $A_{i,j}^{\otimes n}$ and $\mathcal{P}^n(i, j, G)$ for $(A^{\otimes n} \otimes v)_i$ and $\mathcal{P}^n(i \rightarrow, G)$ in the proofs. This leads to an upper bound in $O((\Delta - \delta) \cdot N^2 / (\varrho - \varrho_{\text{nc}}))$, but gives no hint on the relationships between the transient of a max-plus matrix A , and the transients of the max-plus systems $x_{A,v}$. In this section, we show that up to some constant $B_{\text{m/s}}(G)$, the transient of matrix A is actually equal to the transient of some specific systems $x_{A,v}$ where $\|v\|$ is in $O((\Delta - \delta) \cdot N^2)$. Combined with the general upper bounds on the system transient established in Theorems 5, 6 or 8, this result gives upper bounds on the matrix transient, each of which is also in $O((\Delta - \delta) \cdot N^2 / (\varrho - \varrho_{\text{nc}}))$.

First, we derive a general property of strongly connected graphs from the definition of exploration penalty.

Lemma 24. *Let G be a strongly connected graph. For any pair of nodes i, j of G and any integer $n \geq \text{ep}(G) + c(G) + cd(G) - 1$, there exists a path π from i to j such that $n - \ell(\pi) \in \{0, \dots, c(G) - 1\}$.*

Proof. Let i, j be any two nodes, and let π_0 be a simple path from i to j . For any integer n , consider the residue r of $n - \ell(\pi_0)$ modulo $c(G)$. By definition of $\text{ep}(G)$, if $n - \ell(\pi_0) - r \geq \text{ep}(G)$, then there exists a closed path γ starting at node j with length equal to $n - \ell(\pi_0) - r$. Then, $\pi_0 \cdot \gamma$ is a path from i to j with length $n - r$, where $r \in \{0, \dots, c(G) - 1\}$. The lemma follows since $n - \ell(\pi_0) - r \geq \text{ep}(G)$ as soon as $n \geq \text{ep}(G) + cd(G) + c(G) - 1$. \square

Lemma 25. *Let $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ irreducible, $G = G(A)$, and let n be any integer such that $n \geq \text{ep}(G) + c(G) + cd(G) - 1$. Then $A_{i,j}^{\otimes n+c(G)} = -\infty$ if and only if $A_{i,j}^{\otimes n} = -\infty$.*

Proof. It is equivalent to claim that $\mathcal{P}^{n+c(G)}(i, j, G) = \emptyset$ if and only if $\mathcal{P}^n(i, j, G) = \emptyset$ for any integer $n \geq \text{ep}(G) + c(G) + cd(G) - 1$.

Suppose $\mathcal{P}^{n+c(G)}(i, j, G) \neq \emptyset$, and let $\pi_0 \in \mathcal{P}^{n+c(G)}(i, j, G)$. By Lemma 24, there exists a path $\pi \in \mathcal{P}(i, j, G)$ such that $n = \ell(\pi) + r$ with $r \in \{0, 1, \dots, c(G) - 1\}$. Lemma 15 implies that $c(G)$

divides $\ell(\pi_0) - \ell(\pi) = (n + c(G)) - (n - r) = c(G) + r$; hence $c(G)$ divides r . Therefore, $r = 0$, i.e., $\ell(\pi) = n$ and thus $\mathcal{P}^n(i, j, G) \neq \emptyset$.

The converse implication is proved similarly. \square

We now define the matrix/system bound for graph G by

$$B_{m/s}(G) = 2cd(G) + ep(G) + \max_{H \in \mathcal{C}(G_c)} c(H) + \max_{H \in \mathcal{C}(G_c)} ep(H) - 1$$

and let

$$\mu(A) = \sup \left\{ A_{i,k}^{\otimes n} - A_{i,j}^{\otimes n} \mid i, j, k \in V, n \geq B_{m/s}(G(A)), A_{i,j}^{\otimes n} \neq -\infty \right\}.$$

Obviously, $\mu \in \mathbb{R}_{\max}$.

In the following lemma, we fix a node $j \in V$, and a vector $v \in \mathbb{R}^N$ such that

$$\forall k \in V \setminus \{j\}, \quad v_j - v_k \geq \mu. \quad (18)$$

Such a vector v exists, and we let $x = x_{A,v}$.

Lemma 26. *Let $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ irreducible and $G = G(A)$. For any integer $n \geq B_{m/s}(G)$, any node i , and any positive integer p that is a multiple of $c(G)$, if $x_i(n+p) = x_i(n)$, then $A_{i,j}^{\otimes n+p} = A_{i,j}^{\otimes n}$.*

Proof. Let i be any node in G , and n be any integer such that $n \geq B_{m/s}(G)$. Since $B_{m/s}(G) \geq ep(G) + c(G) + cd(G) - 1$, and $c(G)$ divides p , we derive from Lemma 25 that $A_{i,j}^{\otimes n+p} = -\infty$ if and only if $A_{i,j}^{\otimes n} = -\infty$. There are two cases to consider:

1. $A_{i,j}^{\otimes n} = -\infty$ and $A_{i,j}^{\otimes n+p} = -\infty$. In this case, $A_{i,j}^{\otimes n+p} = A_{i,j}^{\otimes n}$ trivially holds.
2. $A_{i,j}^{\otimes n} \neq -\infty$, and $A_{i,j}^{\otimes n+p} \neq -\infty$. Recall that

$$x_i(n) = \max \{ A_{i,k}^{\otimes n} + v_k \mid k \in \{1, \dots, N\} \}.$$

By definition of μ and v , for any node $k \neq j$,

$$A_{i,k}^{\otimes n} - A_{i,j}^{\otimes n} \leq \mu \leq v_j - v_k.$$

Since the latter inequalities trivially hold for $k = j$, it follows that

$$x_i(n) = A_{i,j}^{\otimes n} + v_j.$$

As $n + p \geq n$, we also have

$$A_{i,j}^{\otimes n+p} = x_i(n+p) - v_j = x_i(n) - v_j = A_{i,j}^{\otimes n}.$$

Thus $A_{i,j}^{\otimes n+p} = A_{i,j}^{\otimes n}$ holds in this case.

The lemma follows in both cases. \square

Lemma 27. *Let $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ be an irreducible max-plus matrix, and let $G = G(A)$. Then*

$$\mu(A) \leq \overline{\Delta}(G) \cdot cd(G) - \overline{\delta}(G) \cdot (2cd(G) + ep(G) + \max_{H \in \mathcal{C}(G_c)} c(H) - 1).$$

Proof. First observe that

$$\mu(A) = \mu(\overline{A}) .$$

Hence each term in the inequality to show is invariant under substituting G by \overline{G} , and we may assume without loss of generality that $\varrho(A) = 0$. It follows that

$$\delta(G) \leq 0 \leq \Delta(G) .$$

Then we prove that

$$A_{i,k}^{\otimes n} \leq \Delta(G) \cdot cd(G) . \quad (19)$$

If $A_{i,k}^{\otimes n} = -\infty$, then (19) trivially holds. Otherwise, $A_{i,k}^{\otimes n} = w_*(\hat{\pi})$ for some path $\hat{\pi} \in \mathcal{P}^n(i, k, G)$, and

$$w_*(\hat{\pi}) \leq w_*(\text{Simp}(\hat{\pi})) \leq \Delta(G) \cdot cd(G) .$$

We now give a lower bound on $A_{i,j}^{\otimes n}$ in the case that it is finite, i.e., if $\mathcal{P}^n(i, j, G) \neq \emptyset$. Let u be a critical node, in critical component H , with minimal distance from i and let π_1 be a shortest path from i to u . Further, let π_2 be a shortest path from u to j . Let r denote the residue of $n - \ell(\pi_1 \cdot \pi_2) - ep(G)$ modulo $c(H)$, and let $t = n - \ell(\pi_1 \cdot \pi_2) - ep(G) - r$. Since $t \equiv 0 \pmod{c(H)}$, and

$$t \geq B_{m/s} - 2cd(G) - ep(G) - c(H) + 1 \geq ep(H) ,$$

there exists a closed path γ_c of length t in component H starting at node u . Let $s = ep(G) + r$; then, $s \geq ep(G)$. Moreover, $s = n - \ell(\pi_1 \cdot \gamma_c \cdot \pi_2)$, and $\pi_1 \cdot \gamma_c \cdot \pi_2 \in \mathcal{P}(i, j, G)$. By Lemma 15, it follows that $c(G)$ divides s . Hence there exists a closed path γ_{nc} of length s starting at node j .

Now define $\pi = \pi_1 \cdot \gamma_c \cdot \pi_2 \cdot \gamma_{nc}$. Figure 5 shows path π . Clearly, $\ell(\pi) = n$ and

$$w_*(\pi) \geq \delta \cdot (n - t) \geq \delta \cdot (2cd(G) + ep(G) + c(H) - 1) ,$$

and so

$$A_{i,j}^{\otimes n} \geq \delta \cdot (2cd(G) + ep(G) + \max_{H \in \mathcal{C}(G_c)} c(H) - 1) . \quad (20)$$

Combining (19) and (20) concludes the proof. \square

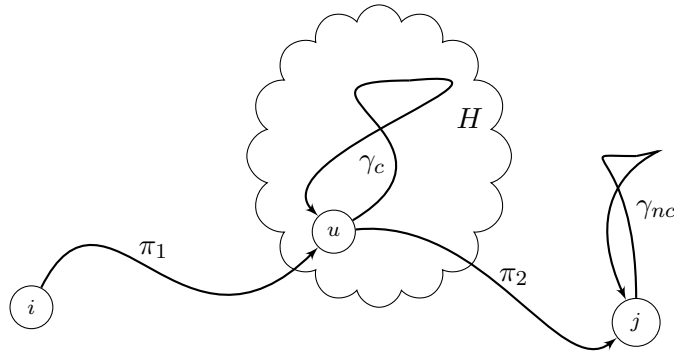


Figure 5: Path π in proof of Lemma 27

From Lemma 27 with $cd(G) \leq N - 1$ and $c(H) \leq N$ we immediately obtain:

Corollary 6. *Let $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ be an irreducible max-plus matrix, and let $G = G(A)$. Then*

$$\mu(A) \leq (\Delta(G) - \delta(G)) \cdot (N - 1) + (\varrho(G) - \delta(G)) \cdot (2(N - 1) + ep(G)) .$$

For an irreducible matrix $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ and j , with $1 \leq j \leq N$, we define vector v^j by:

$$\forall k \in V, \quad v_k^j = \begin{cases} 0 & \text{if } k = j \\ -\mu(A) & \text{otherwise} . \end{cases}$$

Theorem 9. *Let $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ be an irreducible max-plus matrix and $G = G(A)$. Then*

$$n_A \leq \max\{B_{m/s}(G) , n_{A,v^1} , \dots , n_{A,v^N}\} .$$

Proof. We easily check that each vector v^j satisfies condition (18), and $\|v^j\| = \mu(A)$. Since $c(G)$ divides $c(A)$, we can apply Lemma 26 with all vectors v^j and $p = c(A)$. Then, we obtain

$$\forall n \in \mathbb{N} : n \geq \max\{B_{m/s}(G) , n_{A,v^1} , \dots , n_{A,v^N}\} \implies n \geq n_A ,$$

which shows

$$n_A \leq \max\{B_{m/s}(G) , n_{A,v^1} , \dots , n_{A,v^N}\} .$$

□

Hence up to $B_{m/s}$, the transient of an irreducible matrix A is equal to the transient of some of the systems (A, v^j) . Interestingly, this result could be compared to the equality

$$n_A = \max\{n_{A,e^1} , \dots , n_{A,e^N}\}$$

established in Section 2.5, where e^j denotes the vector in \mathbb{R}_{\max}^N which is similar to vector v^j except the j th component equals to $-\infty$ instead of $-\mu(A)$.

Combination of Theorem 9 and Corollary 6 finally yields:

Corollary 7. *Let $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ be an irreducible max-plus matrix and $G = G(A)$. Then*

$$n_A \leq \max\{3(N - 1) + ep(G) + \max_{H \in \mathcal{C}(G_c)} ep(H) , B(G)\} ,$$

where $B(G)$ is the minimum of the bounds stated in Corollaries 3, 4, and 5, with $\|v\|$ replaced by $(\Delta(G) - \delta(G)) \cdot (N - 1) + (\varrho(G) - \delta(G)) \cdot (2(N - 1) + ep(G))$.

8 Discussion

In this section, we discuss the relation to previous work on system and matrix transients and show how to apply our results to the analysis of the Full Reversal algorithm. In particular, we also obtain a new result on the transient of Full Reversal scheduling on trees.

8.1 Relation to previous work

8.1.1 Even and Rajsbaum

Even and Rajsbaum [4] proved an upper bound on the transient of $x_{A,v}$ for an irreducible matrix $A \in \mathcal{M}_{N,N}(\mathbb{N} \cup \{-\infty\})$ and a vector $v \in \mathbb{N}^N$. With our notation and $G = G(A, v)$, their bound reads

$$n_{A,v}^{\text{ER}} = l_0(G) + N + 2N^2, \quad (21)$$

where $l_0(G)$ is an upper bound on the length n of maximum weight paths that contain only non-critical nodes. It thus corresponds to our $B_c(G)$ bound, and is given by,

$$l_0(G) = \frac{N}{f(G)} \left(\|v\| + (\Delta - \delta)(N - 1) \right) + (N - 1),$$

where $f(G)$ is defined by

$$f(G) = \inf \left\{ \ell(\gamma) \varrho(G) - w_*(\gamma) \mid \gamma \in \mathcal{P}_\circ(G) \wedge \gamma \text{ is non-critical} \right\}.$$

Since a path that has no critical nodes is non-critical, this can be bounded by

$$f(G) \leq N \cdot (\varrho(G) - \varrho_{\text{nc}}(G)).$$

Together with Corollary 1 it thus follows that,

$$l_0(G) \geq \frac{\|v\| + (\Delta - \delta) \cdot (N - 1)}{\varrho - \varrho_{\text{nc}}} + (N - 1) \geq B_c(G). \quad (22)$$

The $N + 2N^2$ term in (21) corresponds to the second term in the maximum of our explorative and repetitive bounds. Even and Rajsbaum extend realizers that contain critical nodes by adding a *spanning Eulerian derivative* (SED) of length $O(N^2)$ for each critical component visited by the original realizer. An SED is a closed path that visits each node in the critical component. They add SEDs because of the way they reduce paths: Their *non-critical part* construction may disconnect the original path by removing collections of critical edges that may be combined to a critical closed path. This is a major difference to our approach: While we, too, reduce realizers, our construction does not disconnect paths.

With the SED construction Even and Rajsbaum can pump the length of the constructed path in multiples of $c(G_c)$, by adding paths from the set of *all* elementary closed paths of *all* visited critical component, while still maintaining the property of being a realizer. The major difference to our approach here is that we pump the reduced realizers' length with either the (i) explorative method, or (ii) the repetitive method such that the resulting paths remain realizers, where the idea of both (i) and (ii) is that we extend the reduced realizers by adding a closed path at a *single* critical node whose subpaths are from a *small restricted set* of elementary closed paths of the node's critical component. In case of (ii) we even restrict the set of elementary closed paths, which are used for pumping, to a *single* elementary closed path the critical node lies on.

Not resorting to the SED construction, we obtain upper bounds in which the critical path contribution may be linear in the number of nodes (cf. Section 8.3 below).

8.1.2 Hartmann and Arguelles

Hartmann and Arguelles [9] state the following bounds

$$n_{A,v}^{\text{HA}} = \max \left\{ \frac{\|v\| + N \cdot (\Delta - \delta)}{\varrho - \varrho^0}, 2N^2 \right\}, \quad (23)$$

$$n_A^{\text{HA}} = 2N^2 \cdot \frac{\Delta - \delta}{\varrho - \varrho^0}, \quad (24)$$

where $\varrho^0(G(A))$ is defined on the max-balanced reweighted graph [16] of graph $G(A)$, in the following called $G_{\max}(A)$, as the supremum of $\varrho' \in \mathbb{R}_{\max}$ such that the subgraph of $G_{\max}(A)$ induced by the edge set of edges of $G_{\max}(A)$ with weight at least ϱ' has a strongly connected component with only non-critical nodes. In case $\varrho^0 = -\infty$, they set $\varrho - \varrho^0 = \Delta(G) - \delta(G)$. Note that the first term in (23) corresponds to our $B_c(G)$ bound, however, is incomparable with it in general. Hartmann and Arguelles reduce a realizer similar to Even and Rajsbaum, i.e., the reduced path may be disconnected. To establish connectedness, they add at most N previously removed closed paths. They then add additional critical closed paths to arrive at the right residue class modulo $c(G_c)$. We have shown in Section 6.2 that this last step is, in fact, unnecessary and that the term $2N^2$ in $n_{A,v}^{\text{HA}}$ is therefore improvable to N^2 . A major difference of our bounds to (23) and (24) is that (23) and (24) cannot become linear in N .

8.1.3 Soto y Koelemeijer

Soto y Koelemeijer [11] presented an upper bound on both the transient of system $x_{A,v}$, in the following denoted by $n_{A,v}^{\text{SyK}}$, and the transient of matrix A , denoted by n_A^{SyK} , for irreducible $A \in \mathcal{M}_{N,N}(\mathbb{R}_{\max})$ and $v \in \mathbb{R}^N$. With our notation they read,

$$n_{A,v}^{\text{SyK}} = \max \left\{ \frac{\|v\| + N \cdot (\Delta - \delta)}{\varrho - \varrho_1}, 2N^2 \right\}, \quad (25)$$

$$n_A^{\text{SyK}} = \max \left\{ N^2 \cdot \frac{\Delta - \delta}{\varrho - \varrho_1}, 2N^2 \right\}, \quad (26)$$

where, by setting $G = G(A)$,

$$\varrho_1(G) = \sup \left\{ \frac{w_*(\gamma)}{\ell(\gamma)} \mid \gamma \in \mathcal{P}_{\circ}(G) \wedge \gamma \text{ is non-critical} \right\}.$$

We start our comparison with the bound $n_{A,v}^{\text{SyK}}$. By the argument that a path that has no critical nodes is non-critical, we obtain, $\varrho_1(G) \geq \varrho_{\text{nc}}(G)$, and thereby,

$$\varrho(G) - \varrho_1(G) \leq \varrho(G) - \varrho_{\text{nc}}(G). \quad (27)$$

Thus the first term in (25) is greater or equal to $B_c(G)$. In contrast to the bounds stated in Theorems 5, 6 and 8, the term $2N^2$ in (25) prevents the bound from potentially becoming linear in N . Further Corollary 5 shows that our upper bound is strictly less than the upper bound in (25).

8.1.4 Matrix vs. system transients

In contrast to Soto y Koelemeijer [11] and Hartmann and Arguelles [9], we proved the bound on the matrix transient n_A by *reduction* to the system transient $n_{A,v}$, for properly chosen v , shedding some light on how matrix and system transient relate. With this method we obtain the following bound from Corollaries 7 and 5,

$$n_A \leq \max \left\{ (N-1) \cdot \frac{2(\Delta - \delta) + (\varrho - \delta)(N+2)}{\varrho - \varrho_{\text{nc}}}, N^2 \right\},$$

which is strictly better than the bounds n_A^{HA} and n_A^{SyK} with respect to the second term, however, in general is incomparable to it with respect to the first term. By conservatively bounding $\varrho \leq \Delta$, we see that our resulting bound on n_A is close to the bound n_A^{SyK} by Soto y Koelemeijer. Note that, however, our bound can become linear in N , since both $B_{\text{m/s}}$ and our bounds on $n_{A,v}$ can become linear in N .

8.2 Integer matrices

In the case that all weights of the nontrivial strongly connected e-weighted graph G are integers, we can derive a lower bound on the term $\varrho(G) - \varrho_{\text{nc}}(G)$, which appears in all our upper bounds. Let N be the number of nodes in G . We show that $1/(\varrho(G) - \varrho_{\text{nc}}(G))$ is in $O(N^2)$. This statement is trivial if $\varrho_{\text{nc}}(G) = -\infty$; so we assume that $\varrho_{\text{nc}}(G)$ is finite.

Let $\varrho(G) = x/y$ where x and y are coprime integers and y is positive. Then for every critical closed path γ , we have $w_*(\gamma)/\ell(\gamma) = x/y$, i.e., $y \cdot w_*(\gamma) = \ell(\gamma) \cdot x$. Because x and y are coprime, this implies that y divides $\ell(\gamma)$. Hence y divides all critical closed path lengths, i.e., it divides their greatest common divisor $\gcd\{c(H) \mid H \in \mathcal{C}(G_c)\}$. In particular,

$$y \leq \gcd\{c(H) \mid H \in \mathcal{C}(G_c)\}.$$

If $\varrho_{\text{nc}}(G) = z/u$ where z and u are coprime integers and u is positive, then necessarily $u \leq cr_{\text{nc}}(G)$, where $cr_{\text{nc}}(G)$ is the maximum path length of elementary closed paths whose nodes are non-critical. Because $\varrho(G) - \varrho_{\text{nc}}(G) > 0$, we have

$$\varrho(G) - \varrho_{\text{nc}}(G) = \frac{x \cdot u - z \cdot y}{y \cdot u} \geq \frac{1}{y \cdot u},$$

which implies, by combining the above inequalities, that

$$\frac{1}{\varrho(G) - \varrho_{\text{nc}}(G)} \leq y \cdot u \leq \gcd\{c(H) \mid H \in \mathcal{C}(G_c)\} \cdot cr_{\text{nc}}(G) \leq (N - N_{\text{nc}}) \cdot N_{\text{nc}} \leq \frac{N^2}{4}. \quad (28)$$

Combination with our bounds on $n_{A,v}$, for constant $\|v\|$ and $\Delta - \delta$, thus yields $n_{A,v} = O(N^3)$.

8.3 Full Reversal routing and scheduling

Full Reversal is a simple algorithm on directed graphs used in routing [6] and scheduling [7]. It comprises only a single rule: Each sink reverses all its (incoming) edges. Given an initial graph G_0 with N nodes, we define a *greedy execution* of Full Reversal as a sequence $(G_t)_{t \geq 0}$ of graphs, where G_{t+1} is obtained from G_t by reversing the edges of all sinks in G_t . As no two sinks in G_t can

be adjacent, G_{t+1} is well-defined. For each $t \geq 0$ we define the *work vector* $W(t)$ by setting $W_i(t)$ to the number of reversals of node i until iteration t , i.e., the number of times node i is a sink in the execution prefix G_0, \dots, G_{t-1} .

Charron-Bost et al. [5] have shown that the sequence of work vectors can be described as a min-plus linear dynamical system: Define a *min-plus* matrix as a matrix with entries in $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$. We, analogously to max-plus, define the matrix multiplication

$$(A \otimes' B)_{i,j} = \min\{A_{i,k} + B_{k,j} \mid 1 \leq k \leq N\} .$$

It is $A \otimes' B = -((-A) \otimes (-B))$, where $(-M)_{i,j}$ is $-M_{i,j}$ for matrix M . Generalizing a result by Charron-Bost et al. [5, Corollary 2], we obtain

$$W(0) = (0, \dots, 0) \quad \text{and} \quad -W(t+1) = (-A) \otimes (-W(t)) ,$$

where

$$A_{i,j} = \begin{cases} 0 & \text{if } (j, i) \text{ is an edge of } G_0 \\ 1 & \text{if } (j, i) \text{ is not an edge of } G_0 \text{ and } (i, j) \text{ is an edge of } G_0 \\ +\infty & \text{otherwise} . \end{cases}$$

We distinguish two cases that differ in the initial graph G_0 : using Full Reversal as a routing algorithm, and using Full Reversal as a scheduling algorithm.

8.3.1 Full Reversal routing

In the routing case, the initial graph G_0 contains a nonempty set of *destination nodes*, which are characterized by having a self-loop. The initial graph without these self-loops is required to be weakly connected and acyclic. It was shown that for such initial graphs, the execution terminates (eventually all G_t are equal), and after termination, the graph is destination-oriented, i.e., every node has a path to some destination node [5, 6].

In the following we specialize our bounds on the transient of a linear max-plus system to obtain bounds on the transient of $(W(t))_{t \geq 0}$, i.e., on the termination time $\theta(G_0)$ of greedy Full Reversal routing executions starting from initial graph G_0 . For that we define en-weighted graph $G = G(-A, 0)$ for the max-plus matrix $-A$ obtained from initial graph G_0 . Because $-A_{i,i} = 0$ if i is a destination node, $\varrho(G) = 0$. The set of critical nodes in G is equal to the set of destination nodes in G_0 and each critical component of G is trivial. Observe that $-A$ is an integer max-plus matrix with $\Delta_{\text{nc}} \leq 0$ and $\delta = -1$. Hence $\varrho_{\text{nc}}(G) \leq -1/N_{\text{nc}} \leq -1/(N-1)$, where N_{nc} is the number of non-critical nodes in G . By Corollary 1,

$$B_c(G) \leq (N-1)^2 .$$

Since for the critical components H of G , $c(H) = g(H) = 1$, we obtain from Theorem 3 and Corollary 2, an upper bound on the termination time of Full Reversal routing starting from initial graph G_0 , for $N \geq 3$, is

$$\theta(G_0) \leq (N-1)^2 ,$$

since the second term in the maximum in $B_e(G)$ is at most $2(N-1)$.

If the undirected support of initial graph G_0 without the self-loop at the destination nodes is a *tree*, we can use our bounds to give a new proof that the termination time of Full Reversal routing

is linear in N [5, Corollary 5]. In that case it holds that $\varrho(G) = 0$, and either $\varrho_{\text{nc}}(G) = -1/2$ or $\varrho_{\text{nc}}(G) = -\infty$. In both cases we obtain from Corollary 1,

$$B_c(G) \leq 2(N - 1) .$$

In the same way as above, we obtain from Theorem 3 and Corollary 3,

$$\theta(G_0) \leq 2(N - 1) ,$$

which is linear in N .

8.3.2 Full Reversal scheduling

When using the Full Reversal algorithm for scheduling, the undirected support of the weakly connected initial graph G_0 is interpreted as a conflict graph: nodes model processes and an edge between two processes signifies the existence of a shared resource whose access is mutually exclusive. The direction of an edge signifies which process is allowed to use the resource next. A process waits until it is allowed to use all its resources—that is, it waits until it is a sink—and then performs a step, that is, reverses all edges to release its resources. To guarantee liveness, the initial graph G_0 is required to be acyclic.

Contrary to the routing case, critical components have at least two nodes, because there are no self-loops. But it still holds that $-A$ is an integer max-plus matrix with $\Delta_{\text{nc}} \leq 0$ and $\delta = -1$. Hence, by using (28) and Corollary 1, we get

$$B_c(G) \leq \frac{N^2(N - 1)}{4} ,$$

which, by using either Corollary 3, 4, or 5 shows that the transient for Full Reversal scheduling is at most cubic in the number N of processes. This is an improvement over a result by Malka and Rajsbaum [17, Theorem 6.4] who proved that the transient is at most in the order of N^4 .

In the case of Full Reversal scheduling on *trees*, it holds that $\varrho = -1/2$, and $\varrho_{\text{nc}} = -\infty$. Thus by Corollary 1, $B_c(G) = 0$. Further, $G_c = G$ and $c(G) = d(G) = g(G) = 2$. Theorem 3 and Corollary 3 then imply,

$$B_e(G) \leq 6(N - 3) .$$

The resulting upper bound on the transient of Full Reversal scheduling on trees is hence linear in N , which was previously unknown.

Acknowledgments

The authors would like to thank Sergey Sergeev and François Baccelli for providing useful references.

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